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UNIVERSITY OF ALBERTA

SPHERICAL HARMONIC ANALYSIS OF THE ELECTROENCEPHALOGRAM

**BY
GREGORY RIDLEY SHAW**

**A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE
OF
DOCTOR OF PHILOSOPHY**

DEPARTMENT OF ELECTRICAL ENGINEERING

**EDMONTON, ALBERTA
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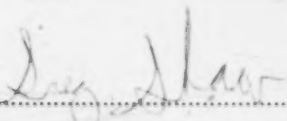
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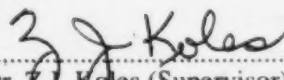
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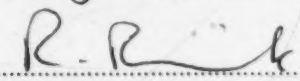

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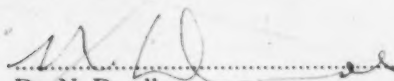
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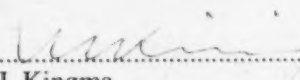
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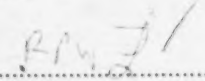
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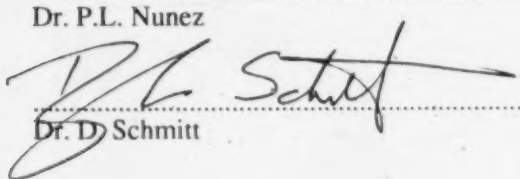

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Abstract

Spatial harmonic analysis of the electroencephalogram (EEG) is complicated by the curvature of the scalp. One approach to account for this curvature is to perform a spherical harmonic (SH) analysis of the EEG. The purposes of this research were to test the validity of the SH expansion and to perform a SH analysis of the EEG. Experimental results showed that the most reliable estimates of the SH coefficients were obtained for a first degree expansion. Lissajous trajectories of the first degree SH coefficient sequences illustrated that planar elliptical trajectories sometimes formed during periods of strong alpha activity. This latter result suggests that a portion of the alpha activity consists of either standing or traveling waves.

Harmonic analysis of the spatial EEG raised the question about the adequacy of the sampling. Experimental results revealed that the spatially aliased variance of the signal was related to the temporal frequency, with minimal aliasing occurring from 5 to 12 Hz and with significantly increased aliasing beyond 12 Hz. However, these results also indicated that the spatial aliasing of the EEG was limited to an acceptable level.

Interpretation of the results obtained from SH analysis is complicated by the activity of the reference electrode. In order to solve this problem, methods of eliminating the activity of the reference were considered, with special focus on the average reference. Experimental results illustrated that the average reference increases the temporal variance of perimeter channels relative to the variance of centrally located channels. In order to reduce this effect, a new reference was presented that is based on the average activity of the perimeter electrodes.

One application of the SH expansion of the EEG is to produce smoothed topographic maps of the spatial activity. This application raised the question as to the optimality of these maps. The quality of the SH maps were compared, using the method of cross-validation, with the maps produced by the bilinear and spherical spline models. Experimental results showed that the spherical spline maps were similar to the optimal maps while the bilinear and SH expansion maps had a significantly increased error level.

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Table of Contents

1 Introduction	1
References	4
2 EEG Theory and Data Acquisition	6
2.1 Anatomy and Physiology of the Brain	6
2.2 Analysis of the EEG	8
2.2.1 Temporal Analysis	8
2.2.2 Spatial Analysis	9
2.2.3 Local and Global Models	11
2.3 Data Acquisition System and Data Preprocessing	11
2.4 EEG Data Base and Artifact Editing	17
References	19
3 Spherical Harmonic Analysis	21
3.1 Spherical Harmonic Expansion of a Spherical Function	21
3.2 Properties and Applications of Spherical Harmonic Expansions	26
3.2.1 Multipole Expansion Equivalence	26
3.2.2 Rotation Invariance	27
3.2.3 Relation to Potential Theory	27
3.2.4 Spatial Laplacian	28
3.2.5 Spatial Power Spectrum	29
3.3 Spherical Harmonic Expansion of an Incomplete Function	30
3.4 Spherical Harmonic Expansion of a Sampled Function	34
References	38
4 Reference Problem	39
4.1 Reference Free Transformations	40
4.2 Average Reference	42
4.2.1 Analysis via Planar Approximation	43
4.2.2 Analysis via Spherical Approximation	47
4.2.3 Additional Problems with the Average Reference Method	48
4.3 Perimeter Reference	49
4.4 Experimental Results	52
4.5 Discussion of Results	63
4.6 Conclusions	64
References	65
5 Estimating the Spatial Bandwidth of the EEG	66
5.1 Introduction	66
5.2 Data Collection and Preprocessing	67
5.3 Analysis Methods and Results	70
5.3.1 Spatial Spectral Analysis	70
5.3.2 Temporal Spectral Analysis of the Aliased Signal	76
5.4 Discussion of Results	78
5.5 Conclusions	79
References	80

6 Comparison of Several Spatial Models for the EEG	81
6.1 Cross-Validated Models	81
6.2 Spatial Models	82
6.2.1 Bilinear Model	82
6.2.2 Spherical Spline Model	84
6.2.3 Truncated Spherical Harmonic Expansion	85
6.3 Multiple Linear Regression Model	86
6.3.1 Constrained MLR Model	87
6.4 Quality of Fit	88
6.5 Determination of the Electrodes' Spherical Coordinates	90
6.6 Experimental Results	95
6.6.1 Constrained MLR Model	95
6.6.2 Spherical Spline Model	97
6.6.3 Spherical Harmonic Expansion Model	98
6.6.4 Bilinear Model	98
6.6.5 Comparison and discussion of results	99
6.7 Conclusions	110
References	141
7 Spherical Harmonic Analysis of the EEG	112
7.1 Smoothing of the EEG by a Spherical Harmonic Expansion	112
7.1.1 Reduced Rank Models	113
7.1.2 Optimal Reduced Rank Projection	113
7.1.3 Experimental Results	115
7.1.4 Discussion of Results	125
7.2 Analysis of the Spherical Harmonic Coefficient Sequences	127
7.2.1 Experimental Results	127
7.2.2 Discussion of Results	127
7.3 Lissajous Trajectories of the Spherical Harmonic Coefficient Sequences	132
7.3.1 Experimental Results	133
7.3.2 Discussion of Results	143
7.4 Conclusions	143
References	144
8 Summary and Conclusions	145
8.1 Summary of Results	145
8.2 Conclusions	147
8.3 Suggestions for Further Research	148
Appendix A - Legendre Polynomials and Associated Functions	149
Appendix B - Spherical Harmonic Expansion of a Function Defined on the Surface of a Spherical Cap	150
Appendix C - Relationship Between the Sum of the Squares of the Spatial Samples and the Spherical Harmonic Coefficients	151

List of Figures

No.	Title	page
2.1	Theoretical current sheet generator located in the cerebral cortex.	7
2.2	EEG data acquisition system.	14
2.3	Time skewed sampling of the EEG channels.	15
2.4	Electrode montage.	15
2.5	Typical magnitude and phase response of the FIR correction filter.	16
3.1	Spherical coordinate system.	22
3.2	Plot of $P_n(\cos \phi)$ and $P_n^1(\cos \phi)$ vs. ϕ .	24
3.3	Typical plot of the zero-value paths for the zonal, sectorial and tesseral harmonics.	25
3.4	Plot of $R(0, n; 0, q; \phi_0)$ vs. ϕ_0 for (a) $n=0$ and (b) $n=1$ and q ranging from 0 to 5.	33
4.1	Contour plot of $ H_{xy}(k_x, k_y) ^2$ vs. spatial wavenumber for corner and center samples.	45
4.2	Spatial plot of the mean values of σ_i^2 contained in Table 4.3.	53
4.3	Spatial plot of the mean values of σ_i^2 contained in Table 4.4.	54
4.4	Spatial plot of the mean values of σ_i^2 contained in Table 4.5.	55
4.5	Spatial plot of the mean values of σ_i^2 contained in Table 4.6.	56
4.6	Spatial plot of the mean values of σ_i^2 contained in Table 4.7.	57
4.7	Spatial plot of the mean values of σ_i^2 contained in Table 4.8.	58
4.8	Spatial plot of the mean values of NCV_i shown in Table 4.9.	59
4.9	Spatial plot of the mean values of NCV_i shown in Table 4.10.	60
4.10	Spatial plot of the mean values of NCV_i shown in Table 4.11.	61
4.11	Spatial plot of the mean values of NCV_i shown in Table 4.12.	62
5.1	Normal and circular 31-channel montages.	68
5.2	Averaged temporal PSDF estimates for the eyes closed and eyes positioned average referenced epochs.	69
5.3	Normalized spatial PSDF estimates.	73
5.4	$E_1(K)$ vs. K for the average referenced epochs.	74
5.5	Plot of $E_2(6, \alpha)$ and $E_1(6)$ vs. α .	75

5.6	Temporal spectral plots of $E_3(6, w)$.	77
6.1	3-D solid frame representation of the electrodes' actual positions measured from five subjects.	92
6.2	Planar projection of the electrodes' estimated spherical coordinates.	94
6.3	3-D representation of the averaged electrodes' spherical coordinates.	94
6.4	E^2 vs. rank for the constrained MLR model.	96
6.5	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.7.	101
6.6	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.8.	102
6.7	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.9.	103
6.8	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.10.	104
6.9	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.11.	105
6.10	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.12.	106
6.11	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.13.	107
6.12	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.14.	108
6.13	Normalized residual variance vs. frequency for the constrained MLR, spherical spline, bilinear and spherical harmonic expansion models.	109
6.14	Averaged temporal PSDF estimates for the perimeter referenced EC and EP data sets.	110
7.1	Plot of the sampled spherical harmonic functions.	117
7.2	Plot of the first eight principal components obtained from the left-ear referenced EP data set.	118
7.3	Plot of the first eight principal components obtained from the left-ear referenced EC data set.	119
7.4	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 7.3.	120
7.5	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 7.4.	121
7.6	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 7.5.	122
7.7	Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 7.6.	123
7.8	Normalized residual variance vs. frequency for the perimeter referenced data sets for various spherical harmonic expansion degrees.	124
7.9	Temporal PSDF estimates of the spherical harmonic coefficient sequences.	131
7.10	Plot of the original and bandpass filtered EEG epoch.	134
7.11	Plot of the filtered EEG epoch and the spherical harmonic expansion fit.	135

7.12	Plot of the three first degree spherical harmonic coefficient sequences along with the associated strength.	136
7.13	Planar Lissajous trajectory obtained from the spherical harmonic coefficient sequences.	137
7.14	Rotated version of the Lissajous trajectory shown in Figure 7.13.	137
7.15	Lissajous trajectory of the entering and exiting phases of the planar section shown in Figure 7.14.	138
7.16	Linear Lissajous trajectory of the spherical harmonic coefficient sequences.	139
7.17	Spatial plot of the left-ear referenced EEG epoch.	140
7.18	Spatial plot of the filtered EEG data.	141
7.19	Spatial plot of the 1st degree spherical harmonic expansion fit of the filtered EEG epoch.	142

List of Tables

No.	Title	page
2.1	Magnitude and phase characteristics of frequency sampling FIR correction filter.	13
2.2	Mean and standard deviation of the magnitude and phase responses for the original and corrected calibration data.	13
3.1	Table of $R(0, n; 0, q; \pi/2)$ vs. n and q .	31
4.1	Tabulations of $\sigma_{xy}^2(R)$ for an average referenced bandlimited signal limited to the spatial wavenumbers $k_x^2 + k_y^2 \leq R^2$.	48
4.2	Tabulations of $\sigma_{xy}^2(R)$ for a perimeter referenced bandlimited signal limited to the spatial wavenumbers $k_x^2 + k_y^2 \leq R^2$.	53
4.3	Mean and standard deviation of $\sigma_i^2 \times 100$ for the left-ear referenced eyes closed data set.	53
4.4	Mean and standard deviation of $\sigma_i^2 \times 100$ for the left-ear referenced eyes positioned data set.	54
4.5	Mean and standard deviation of $\sigma_i^2 \times 100$ for the average referenced eyes closed data set.	55
4.6	Mean and standard deviation of $\sigma_i^2 \times 100$ for the average referenced eyes positioned data set.	56
4.7	Mean and standard deviation of $\sigma_i^2 \times 100$ for the perimeter referenced eyes closed data set.	57
4.8	Mean and standard deviation of $\sigma_i^2 \times 100$ for the perimeter referenced eyes positioned data set.	58
4.9	Mean and standard deviation of NCV_i for the average referenced eyes closed data set.	59
4.10	Mean and standard deviation of NCV_i for the average referenced eyes positioned data set.	60
4.11	Mean and standard deviation of NCV_i for the perimeter referenced eyes closed data set.	61
4.12	Mean and standard deviation of NCV_i for the perimeter referenced eyes positioned data set.	62
5.1	Estimates of α based on the log-log spatial spectra plots shown in Figure 5.3.	72
6.1	Averaged spherical coordinates (ϕ, θ) , measured in degrees, of the electrodes based on five subjects.	94

6.2	Mean and standard deviation of E^2 and $NRC \times 100$ for the constrained full-rank MLR model based on individual and various combined data sets.	96
6.3	Mean and standard deviation of E^2 and $NRC \times 100$ vs. κ for the spherical spline model with $\lambda = 0$.	97
6.4	Mean and standard deviation of E^2 and $NRC \times 100$ for the spherical harmonic expansion model of degree N .	98
6.5	Mean and standard deviation of E^2 and $NRC \times 100$ for the bilinear model.	98
6.6	Mean and standard deviation of E^2 and $NRC \times 100$ based on the various prediction models.	100
6.7	Mean and standard deviation of $E_i^2 \times 100$ for the constrained MLR model formed and tested on the 45-epoch EC data set.	101
6.8	Mean and standard deviation of $E_i^2 \times 100$ for the constrained MLR model formed and tested on the 45-epoch EP data set.	102
6.9	Mean and standard deviation of $E_i^2 \times 100$ for the EC perimeter referenced data set based on the spherical spline model ($\kappa = 2, \lambda = 0$).	103
6.10	Mean and standard deviation of $E_i^2 \times 100$ for the EP perimeter referenced data set based on the spherical spline model.	104
6.11	Mean and standard deviation of $E_i^2 \times 100$ for the EC perimeter referenced data set based on the bilinear model.	105
6.12	Mean and standard deviation of $E_i^2 \times 100$ for the EP perimeter referenced data set based on the bilinear model.	106
6.13	Mean and standard deviation of $E_i^2 \times 100$ for the EC perimeter referenced data set based on the spherical harmonic expansion model ($N = 2$).	107
6.14	Mean and standard deviation of $E_i^2 \times 100$ for the EP perimeter referenced data set based on the spherical harmonic expansion model ($N = 2$).	108
7.1	Mean and standard deviation of $E^2 \times 100$ for a degree N spherical harmonic expansion and a rank P PC projection based on the left-ear referenced data sets.	116 *
7.2	Mean and standard deviation of $E^2 \times 100$ for a degree N spherical harmonic expansion and a rank P PC projection based on the perimeter referenced data sets.	116
7.3	Mean and standard deviation of $E_i^2 \times 100$ for a 1st degree spherical harmonic expansion of the EC perimeter referenced data set.	120

7.4	Mean and standard deviation of $E_i^2 \times 100$ for a 2nd degree spherical harmonic expansion of the EC perimeter referenced data set.	121
7.5	Mean and standard deviation of $E_i^2 \times 100$ for a 1st degree spherical harmonic expansion of the EP perimeter referenced data set.	122
7.6	Mean and standard deviation of $E_i^2 \times 100$ for a 2nd degree spherical harmonic expansion of the EP perimeter referenced data set.	123
7.7	Mean and standard deviation of the sum of squares of the spherical harmonic coefficients obtained from the perimeter referenced data sets.	130
7.8	Spatial power spectral S_n estimates based on 1st and 2nd degree spherical harmonic expansions of the perimeter referenced data sets.	130
7.9	Spatial power spectral S_n estimates based on 1st and 2nd degree spherical harmonic expansions of the average referenced data sets.	130

List of Symbols and Abbreviations

a_n^m	spherical harmonic coefficient of degree n and order m
$\vec{a}$	vector of spherical harmonic coefficients
α	rolloff rate of the spatial power spectrum
$\vec{A}$	matrix of eigenvectors
$\vec{C}$	matrix of eigenvectors
CSLT	coefficient sequence Lissajous trajectory
DFT	discrete Fourier transform
$E_1(K)$	estimate of the spatially aliased variance
$E_2(K, \alpha)$	estimate of the spatially aliased variance
$E_3(K, \omega)$	estimate of the spatially aliased variance
E_i^2	normalized residual variance of channel i
E^2	normalized residual variance
EC	eyes closed
EEG	electroencephalogram
EP	eyes positioned
$\vec{i}$	identity vector
$\vec{I}$	identity matrix
K	folding wavenumber index
κ	parameter for spherical spline
KLE	Karhunen-Loève expansion
λ	parameter for spherical spline
MLR	multiple linear regression
N_s	number of spatial samples
N_t	number of temporal samples
NCV_i	normalized variance of channel i
NRC	normalized residual correlation
P	point on the surface of a sphere
$P(k)$	spatial power spectrum
$Pn(k)$	normalized spatial power spectrum

$P_n(x)$	Legendre polynomial of degree n
$P_n^m(x)$	associated Legendre function of the first kind of degree n and order m
ϕ	spherical coordinate (tip angle)
PC	principal component
PSDF	power spectral density function
r	radius of sphere
$r(n)$	reference level
$R(m, n; p, q; \phi_0)$	measure of the orthogonality between Y_n^m and Y_q^p over the surface of a spherical cap defined by ϕ_0
S_n	spatial spectrum of degree n
S	surface of the sphere
SVD	singular value decomposition
$\bar{S}$	similarity matrix
$\bar{\Sigma}$	diagonal matrix of singular values
σ_i^2	variance of channel i
SSE	sum of squared errors
θ	spherical coordinate (spin angle)
U_n^m	surface spherical harmonic function of degree n and order m
v	function defined on the surface of a sphere
$v_k(n)$	spatial sample k and temporal sample n
$\vec{v}(n)$	vector of spatial samples
$\vec{v}_k$	vector of temporal samples
$\vec{V}$	matrix of spatial and temporal samples
$\vec{v}_n(n)$	neutral referenced vector
$\vec{Y}$	matrix consisting of the sampled spherical harmonic functions
Y_n^m	normalized surface spherical harmonic function of degree n and order m
$\zeta(x)$	Riemann zeta function

1 Introduction

The human brain is the most complex and least understood organ in the human body. Although significant knowledge has been gathered about the microscopic functioning of the brain, the macroscopic functioning remains relatively unknown. That is, although the smallest unit in the brain, the neuron, is reasonably well understood, the action of billions of interconnected neurons is at best poorly understood.

One approach to studying the mass action of the brain is to analyze recordings of its electromagnetic activity. Recordings of the brain's electrical activity, referred to as the electroencephalogram (EEG), were first obtained over 60 years ago by Hans Berger [1] by measuring the potential between electrodes placed on the scalp. More recently, it has become possible to measure the magnetic field emitted by the brain with the aid of super cooled magnetic field sensors [2] [3]. However, due to the high cost of the magnetic field sensors used to measure the magnetic field, it is still preferred to record and analyze the potential measurements.

Hans Berger believed that the EEG would provide a window on the functioning of the brain. After many decades of research, the EEG is mainly used as a clinical tool [4] since it has been found that specific waveforms in the EEG are often correlated to pathologic conditions. For example, the EEG can be used to detect and localize abnormal brain functioning, as in epilepsy [5], to detect the presence of intracranial tumors, and to monitor the effects of medications [6] on patients. Abnormalities in lower brain functioning can be detected by measuring the brain's evoked response via the EEG [7]. The EEG has also found applications as a research tool to study the functioning of the human brain [8].

Recently, the clinical application of the EEG has decreased due to the introduction of the modern imaging methods, such as computed tomography (CT) [9], magnetic resonance imaging (MRI) [10], positron emission tomography (PET) [11], and single-photon emission computed tomography (SPECT) [12]. CT and MRI imaging provide precise information about the anatomical structure of the brain and are extremely useful in localizing physical abnormalities such as tumors. However, these two methods do not convey information about the functional activity of the brain. PET and SPECT imaging provides information about the biochemical functioning of the brain, although the temporal resolution is on the order of minutes.

Thus, despite the recent advances in imaging methods, the EEG still provides the best temporal resolution of the brain's dynamics. A further advantage of the EEG is the minimal cost of the hardware required to acquire the signal, as compared to the imaging methods described in the previous paragraph, allows the EEG to be routinely used as a clinical tool. In addition, the non-invasive procedure for recording the EEG is an important aspect when studying the functioning of the human brain.

In order to extract further information from the EEG, the analysis of the EEG has evolved from the visual inspection of the printed records towards a more quantitative approach. The quantitative analysis of the EEG has proven to be an extremely challenging problem. Multichannel EEG recordings produce large quantities of data, especially when the subject is monitored for long periods of time. However, by far the greatest problem with the quantitative analysis of the EEG is a lack of understanding of the brain's electrophysiology. Since the relationship between the functioning of the brain and the EEG is still largely unknown, it is difficult to develop appropriate mathematical analysis tools.

The EEG contains temporal and spatial information. The temporal component of the EEG is reflective of the brain's dynamics but the relationship poorly understood. Traditional approaches, such as time series analysis, have been employed and have achieved only limited success. Although recent work has shown that at least some of the temporal activity of the brain appears to be chaotic [13] [14], the significance of these results is still unknown. Due to the lack of a quantitative dynamical model of the brain's activity, statistical methods are commonly employed. Fortunately the origin and significance of the spatial signal is better understood. It is assumed that active regions in the brain form current generators and through the effects of volume conduction, a measurable potential forms on the scalp [15].

One approach to analyzing the EEG is to perform a spatial harmonic analysis. Harmonic analysis of experimental data often provides further insight into the properties of the signal. Previous work has assumed that the EEG is sampled on a planar surface and has either used the 2-D DFT of the potential distribution [16] [17] or the maximum entropy method applied to the rms potential distribution [18] as estimators of the spatial spectrum. However, one cannot ignore the fact that the EEG is recorded on the curved surface of the scalp and that projecting the electrodes' positions onto a plane distorts the spatial relationships between electrodes. A more critical problem with a planar analysis is related to the approximately closed surface of the head in which the scalp is embedded. This closed surface imposes a 2-D periodicity in the potential function which cannot be adequately replicated by a planar surface.

The purpose of this research was to perform a spherical harmonic analysis of the spatial EEG. Since the scalp's surface can be reasonably approximated by the surface of a sphere [19], the EEG can be considered as samples of a spherical potential function. Harmonic analysis of spherically sampled data leads to a spherical harmonic analysis, as in the analysis of the earth's magnetic field [20] [21]. Results based on a spherical harmonic analysis of the EEG have not been published in the literature.

Chapter three considers the spherical harmonic expansions of functions defined on the surface of a sphere along with two associated problems that are relevant to the EEG. The problem of determining the spherical harmonic expansion of a function that is known only on a portion of the sphere's surface is considered since the EEG is sampled on approximately the surface of a hemisphere. The other problem considered in this chapter is determining the spherical harmonic expansion of a function given only samples of the function.

One factor that complicates the analysis of the EEG is the activity of the reference electrode. The EEG, when subjected to a quantitative analysis, is usually recorded with respect to the potential of a common reference electrode. Unfortunately, the reference site is not electrically neutral and the choice of the reference site will affect the character of the recorded EEG [15]. One method of eliminating the ambiguity caused by the active reference is to use the average reference [22] [23], which is based on the spatial average of the data. Unfortunately, previous research has shown that the average reference eliminates the effects of the reference at the cost of introducing new problems [24] [25].

The reference problem along with the average reference is studied in detail in Chapter four. Several mathematical analysis procedures for testing the adequacy of the reference estimate are outlined. Since both the theoretical and experimental results illustrated a significant problem with the average reference that has not been previously noted, a new reference was defined in an attempt to reduce this effect.

The harmonic analysis of experimental data must consider if the signal is adequately sampled. Chapter five investigates the adequacy of the spatial sampling of the EEG. Although this problem has been discussed in the EEG literature [26], few quantitative results based on actual EEG data have been presented. This chapter attempts to tackle the problem in a comprehensive fashion based on estimates of the aliased portion of the signal's variance. In order to simplify the analysis steps, a set of EEGs was recorded from a circular array of

electrodes. For a given spatial sampling rate, the aliased portion of the signal's variance was determined from estimates of the signal's spatial power spectral density function. In addition, estimates of the temporal spectrum of the spatially aliased signal are presented. These later results are then used to test for the existence of a dispersive relationship in the EEG, a result that has been proposed but not clearly shown [27] [28].

One application of the spherical harmonic expansion of the EEG is to produce smoothed topographic maps of the spatial electrical activity. Topographic maps are used to localize areas of the scalp that overlay active regions of the brain [29]. However, the accuracy of the potential maps along with the optimal method of producing them are still the subject of debate. The purpose of chapter six is thus to determine the adequacy and optimality of several methods of producing topographic maps of the brain's electrical activity. Although this subject has been considered in the literature [30], the new approach presented in this chapter is cross validation [31]. Cross validation allows the adequacy of the maps along with the optimal estimation of model specific parameters to be determined without the requirement for extra recording channels. In order to determine the adequacy of the various mapping techniques, an optimal reference-free linear model for the spatial EEG is developed.

Chapter seven contains the experimental results relating to the spherical harmonic analysis of the EEG. The first section of the chapter considers the validity and usefulness of the spherical harmonic coefficient sequences obtained from the EEG along with a comparison of the sampled spherical harmonic functions to the optimal basis vectors obtained via principal component analysis [32]. The validity of the expansion is tested by calculating the variances of the spherical harmonic coefficient sequences. The temporal spectral characteristics of the sequences are investigated along with a visual inspection of the coefficient's phase space.

The following chapter reviews the origins of the EEG, the data acquisition system used to record the EEG and several data preprocessing steps. The new material presented in this chapter is a method of measuring and correcting the magnitude and phase response differences between the recording channels. The effects of these differences, although small compared to the original signal, were clearly evident on the results presented in this thesis for uncorrected signals. Surprisingly, this problem seems to have been ignored in the EEG literature. Also included in this chapter is a description of the EEG data sets used in the majority of this thesis.

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2 EEG Theory and Data Acquisition

In order to provide insight into the questions addressed in this thesis, some of the properties of the brain and especially of the EEG are reviewed in the first part of this chapter. Also included is a brief summary of the major results obtained from temporal and spatial analysis of the EEG along with several theoretical models. Since there is a significant amount of literature covering these topics, the review is by no means complete. Although it can be stated that although much has been discovered about the functioning of the brain, this knowledge is far from complete. This incomplete knowledge about the brain's functioning is the major complicating factor in the analysis and interpretation of the EEG.

Also included in this chapter is a description of the data acquisition system used to collect the EEG along with several preprocessing steps and artifact editing. The preprocessing steps were required to compensate for several known distortions introduced by the data acquisition system while artifact editing was used to eliminate 'bad' data. The last section describes the EEG data base that was subjected to further analysis in the following chapters of this thesis.

2.1 Anatomy and Physiology of the Brain

The human brain's anatomy and physiological processes have been extensively studied over the last century. As a first step in understanding the operation of the brain, its anatomical structure was investigated. More recently, the availability of advanced measurement techniques has enabled the study of the physiology of the brain. Since there is a vast body of knowledge related to the anatomy and physiology of the brain, only a brief review follows. For more information, the reader can consult [1] and [2].

The human brain consists of several distinct regions, namely the brain stem, thalamus, cerebellum and the cerebrum. The brain stem connects the spinal cord to the rest of the brain. Located above the brainstem is the thalamus which is involved in transmitting sensory input to the cerebrum. Situated behind the brain stem is the cerebellum which is used to control muscle movement. The remaining section of the brain consists of the cerebrum, which controls higher thought processes. It is theorized that most of the electrical activity recorded in the EEG originates from the cerebrum.

The cerebrum consists of two interconnected hemispheres. Each hemisphere is composed of an outer layer of gray matter called the cerebral cortex while the inner region consists of white matter. The cerebral cortex is composed mainly of neurons while the major component of the white matter are interconnects called axons. Neurons can be considered as the main processing units in the brain while the axons interconnect the neurons.

Each neuron has a central cell body (soma), numerous input and output pathways, namely the dendrites and axons respectively and synapses. The dendrites form a local branch like structure radiating from the soma while a single long axon usually emerges from the soma. The end of the axon again tends to branch out and connect to the dendrites of adjacent neurons. Another important component of the neuron are the synapses which occur at the points of connection between neurons and control the transmission of information from one neuron to another. Between the synapses of two adjacent neurons, there exists a small, but functionally significant space called the synaptic gap.

Neurons in the cerebral cortex tend to be orientated perpendicular to the outer surface and form cortical columns of approximately 200-300 μm in diameter. Each cortical column can be considered as an individual processing unit that is highly interconnected to

other columnar units. Most of the connections between the cortical columns are relatively short range. However, longer range corticocortical connections, which travel through the white matter, can connect distant regions of the brain.

When a stimulus arrives from the dendrites and triggers the activity of a neuron, a change in the permeability of the cell membrane causes a temporary diffusion of ionic fluids through the membrane, resulting in a current flow. The changing membrane permeability then propagates along the axon, resulting in a traveling disturbance called an action potential. In order for the action potential to be transmitted across the synaptic gap to an adjacent neuron, biochemical transmitters are released from the output synapses and diffuse to the input synapse of the adjoining neuron. In general, the input synapses of the neuron can either promote or inhibit the formation of an action potential.

When a neuron is triggered, the resulting current flow creates a measurable electric potential in the vicinity of the neuron. The temporal and spatial characteristics of the associated action potential have been widely investigated and are well understood. If a sufficient number of neurons are simultaneously active, the resulting current flow produces a detectable potential on the surface of the scalp. Unfortunately, the EEG bears little resemblance to an action potential and is not well understood.

It is commonly believed that current sheets form on the outer layer of the cerebral cortex, due to the perpendicular arrangement of the cortical neurons [3]. As a first approximation, each active neuron can be modeled by a current dipole. The synchronous activation of many adjacent neurons can then be modeled as many parallel current dipoles, or as a current sheet. Figure 2.1 shows one such theoretical current sheet along with the resulting current flows. Through the effects of volume conduction, a measurable potential forms on the scalp.

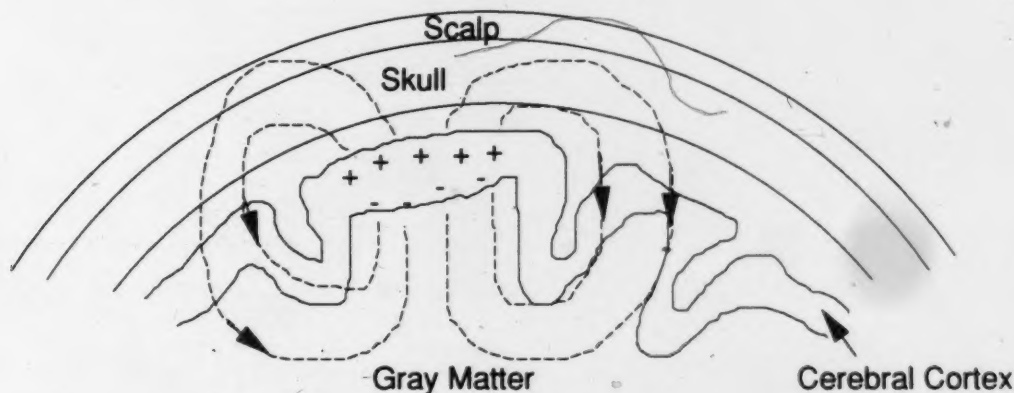


Figure 2.1 - Theoretical current sheet generator located in the cerebral cortex. Outer region consists of the scalp and the skull while the folded region is the cerebral cortex. It is theorized that localized activity in the cerebral cortex forms parallel current dipoles which can be modeled as a current sheet. The extracellular current flow, indicated by the dashed paths, is mainly restricted to the inner regions due to the low conductivity of the skull.

The EEG is usually divided into two main types of activity, namely evoked potentials and background activity. Evoked potentials occur when the subject is provided with an external stimulus, such as a flash of light, which evokes a response from lower brain functioning. Although the evoked response from a single trial is usually dominated by other activity, it can be recovered by the averaging of repeated trials due to repeatability of the signal. In contrast, the background activity of the brain is that activity that the brain itself originates, such as cognitive thought processes.

The complexity of the human brain is due to the large number of neurons ($\sim 10^{10}$) and due to the even larger total number of interconnections between all the neurons ($\sim 10^{14}$). At the present time, there is not sufficient knowledge to accurately model the functioning of the human brain. Thus the problem of understanding the origin and more importantly, the significance of the EEG is a challenging problem.

2.2 Analysis of the EEG

The EEG has been extensively analyzed, both from a theoretical and a clinical perspective [4] [5] [6] [7] [8] [9] [10]. Most of the analysis of the EEG, including the results presented in this thesis, has tended towards describing what has been observed, which has particular significance in a clinical setting. Unfortunately, it is difficult to ascertain the significance of the EEG without a basic understanding of its origin. A more fundamental approach to investigating the EEG is to formulate a mathematical theory of the brain's dynamics. By varying the basic parameters in the model, this could indicate the factors that control the EEG. Basing a theory on a mathematical model is common in other areas of science, especially physics. However, modeling the EEG is still in its infancy due to the extreme complexity of the brain along with incomplete anatomical and physiological information.

The following sections review the major results obtained from a temporal and spatial analysis of the EEG along with several theories about the origin of the electrical activity of the brain.

2.2.1 Temporal Analysis

The EEG has been measured and analyzed for over 60 years and large body of knowledge has been accumulated. Many specific patterns in the EEG have been associated with normal and abnormal brain functioning. Unfortunately, it is not known how the brain actually produces the waveforms and thus the significance of many of the waveforms is not known.

Traditionally, most of the analysis of the EEG has been based on the temporal characteristics of the signal. It has been found that the EEG is composed of many distinct waveforms that depend upon the state and health of the brain. Numerous pathologic conditions of the brain are known to produce characteristic patterns that can be visually detected by trained personnel. Temporal events such as spike activity and sharp waves indicate possible abnormal brain functioning. An epileptic discharge produces a distinct waveform that is easily detected in the EEG.

Another analysis approach is the spectral decomposition of the EEG. Single channel power spectral density function estimates (PSDF) of the EEG are obtained by using traditional signal processing methods, such the averaged DFT or autoregressive models [11]. It is commonly assumed that the EEG can be treated as a linear process with piecewise stationary periods on the order of 10 seconds. The characteristics of the resulting spectrum are then used to classify the type of activity.

The EEG is usually described in terms of the delta (1-4 Hz), theta (4-7 Hz), alpha (7-13 Hz) and beta (13+ Hz) frequency bands. Low frequency waveforms in the delta band, uncommonly observed in healthy subjects, usually indicates a pathologic condition such as a brain tumor. Alpha waves commonly occur when the subject is relaxed and has their eyes closed. Beta band activity can be emphasized by the application of tranquilizers such as diazepam and as well as barbiturates. Frequencies beyond approximately 30 Hz are not commonly considered due to significant contamination from muscle artifacts.

Alpha activity, usually referred to as the alpha rhythm, has been the subject of numerous studies due to its widespread occurrence between subjects and due to the simplicity of the waveform. Approximately 90% of healthy subjects have a detectable alpha rhythm when they are in a relaxed state with their eyes closed. The alpha rhythm appears as a sinusoidal signal that is strongest in the occipital region. Such factors as a change in the mental state and the opening of the subject's eyes attenuate the alpha rhythm. Although the signal characteristics of the alpha rhythm are well-known, the origin and more importantly the significance of the signal is still the subject of debate.

Although the analysis of the EEG by such factors as the frequency content or the presence of known patterns has proven to be clinically useful, this approach has not led to an understanding of the origin of the signal. It is much easier to determine signal properties specific to patients with known pathological conditions than to predict the signal properties based only on a theoretical model of the brain. One should not formulate a theory of brain functioning based solely on experimental data. Instead, experimental data should be used to verify the predictions made from a theoretical model of the brain.

One factor that might partially account for the lack of success in understanding the significance of the EEG is that some of the background activity of the EEG appears to be chaotic [12] [13] [14] [15]. Chaotic processes are by definition extremely unpredictable and non-linear, and can only be parameterized in a very general sense [16]. The description of a chaotic process by traditional signal processing techniques, such as the PSDF, provides little insight into the characteristics of the process. Instead, chaotic processes are better described by such factors as attractors and dimensionality measures.

In general, it can be stated that the origin of the temporal activity of the EEG is not understood at the present time. It has been pointed out by numerous researchers that not even the basic equations, let alone their solutions, are known. More importantly, the significance of much of the activity in the background EEG is unknown.

2.2.2 Spatial Analysis

The EEG also contains spatial information. Both the temporal and spatial aspects of the EEG are routinely considered during an analysis of the EEG. It has been found that many waveforms are commonly observed at specific regions of the scalp, such as the alpha activity being dominant on the occipital region. Also, a localized pathologic condition of the brain, such as a tumor, often produces abnormal EEG activity in the region of the tumor.

In comparison to the temporal activity, the origin of the spatial activity is better understood. It is presently assumed that the EEG activity of the human head is the result of time-varying current generators distributed in a nonhomogeneous conducting medium. The specific distribution of the current generators, which are related to the active regions of the cerebral cortex, define the spatial potential variations in the EEG. Variations in the activity of the cerebral cortex over time results in a time-varying EEG.

The potential distribution on the scalp due to current generators located in the brain can be mathematically modeled by potential theory [17]. Given sufficient anatomical and physiological information, along with the distribution of the current generators, it is possible to solve for the potential distribution on the surface of the head. In potential theory, this problem is referred to as the forward problem and the solution is unique. Although the solution for the potential distribution might not be easy to obtain, at least the basic equations and their properties are well understood.

One complicating factor in modeling the electrical activity of the brain is the nonhomogeneous conductivity of the head. It has been found that the conductivity of the various tissues in the human head varies widely. As a first approximation, the head can be divided into three concentric spherical zones of constant conductivity consisting of the brain, skull and scalp. The conductivity of the scalp and the brain are similar, while the conductivity of the skull is approximately 100 times smaller.

The low conductivity of the skull limits the spatial resolution of the EEG. Signals originating from within the skull are spatially smeared when they are measured on the scalp. Theoretical calculations indicate that the diameter of the point spread function from the outer surface of the cerebral cortex to the scalp is approximately 2 cm [18]. Local features in the potential distribution on the outer surface of the cerebral cortex spaced less than 2 cm apart would then not be clearly observed. Attempts have been made [19] to deconvolve the effects of the skull, but it is difficult to verify the accuracy of the results.

Another area in potential theory is the inverse problem, which can be stated as: given the potential distribution on a closed surface, determine the distribution of the current generators inside that closed surface. Theoretically, the solution to the inverse problem, as applied to the EEG, might indicate which regions of the brain were active at a particular time. Unfortunately, this problem is ill-defined and has no unique solution. That is, there are an infinite number of current generator distributions that can produce the same given surface potential distribution. Also, since the EEG is observed on a non-closed surface, this creates additional problems.

In order to derive a unique solution to the EEG inverse problem, one must specify the type and number of generators along with a physical model for the human head. The spatial EEG is usually described by a three shell model containing several dipole current generators [20] [21]. A more accurate physical description of the head is to use a boundary element model to define the regions of constant conductivity [20].

Given the theoretical model for the electrical activity of the head along with the observed spatial potential distribution, it is possible to estimate the number and location of the equivalent dipole current generators. However, since the solution for the dipole parameters involves non-linear optimization, convergence of the algorithm and the optimality of the solution must be considered. In addition, the positions of the dipoles can be greatly affected by the number of dipoles assumed in the solution. Thus the dipole model is most useful when the EEG activity is restricted to several dipole like generators.

The validity of the three shell dipole model depends mainly on the dipole assumption for the current generators. This assumption is most valid for evoked potential studies, for it is believed that only a few localized regions of the brain are active during the first part of the response. Localized pathologic conditions in the brain, such as tumors, might also be adequately described by dipoles. In contrast the background EEG usually has large regions of the brain active at any given time and thus the dipole model has little justification. In general, it would be unreasonable to expect that a few single dipoles could describe the highly complex activity of the brain.

2.2.3 Local and Global Models

In order to achieve further insight into the brain's electrical activity, local and global models have been proposed [22] [23]. These models attempt to integrate known (and unknown!) physiological information about the brain into a set of mathematical equations. Solving these equations under different conditions then might indicate the origin and significance of the EEG activity.

Local models consider the cortical column as the basic functional unit. The inhibitory and excitatory feedbacks within the unit causes oscillatory behavior in the neurons. Synaptic delay times are considered to be the dominate factor while the propagation time of an action potential along an axon is ignored. The brain is then considered as being composed of many relatively independent functional units, with the signals originating from the units summing together to form the EEG. A major problem with this approach is that the brain's spatial extent is ignored along with the corticocortical connections.

The global model considers the effect of both short and long range corticocortical connections on the activity of the cortical columns. It is postulated that the oscillatory behavior of the cerebral cortex is mainly affected by the propagation velocity of action potentials along the corticocortical axons. It is also postulated that the coupling between different regions of the cerebral cortex results in spatially coherent activity. One important prediction from the global model is the existence of a dispersive relationship between the temporal and spatial waves. A region of the cerebral cortex could trigger a temporal and spatial wavefront, similar to that produced by dropping a stone into a pond of water.

Unfortunately, the present models represent only a preliminary attempt to understand the activity of the brain. Many of the physiological and electrical properties of the brain that are incorporated into the models are still not accurately known. Such factors as biochemical messengers and far electric fields, which are suspected of influencing the EEG, are not included into the models. In addition, it is very likely that additional factors that are not presently known affect the electrical behavior of the brain.

2.3 Data Acquisition System and Data Preprocessing

A block diagram of the EEG data acquisition system is shown in Figure 2.2. Each channel is amplified and bandpassed filtered (0.5-40 Hz.) by the preamplifiers and filters of the Neuroscience Brain Imager system, then digitized and stored by a 32-channel data acquisition unit [24]. The multiplexer samples each channel 120 times per second with the conversion time between channels being 61.5 usec, as shown in Figure 2.3. The digitized values are stored on floppy disks, then later transferred to magnetic tape for permanent storage and further processing.

Figure 2.4 shows the electrode montage. Thirty-one electrodes are mounted on a flexible cap while the reference electrode is placed on the subject's left ear lobe. As a first approximation, the electrode montage forms a 5x7 grid with the four corner positions missing. The remaining channel (channel 32) is used as a event marker.

It is important that the channels' gain and phase responses are matched when attempting a spatial analysis of the data. As each channel has its own set of pre-amplifiers and analogue filters, one would expect some deviation between the channels' magnitude and phase characteristics. Since there is only one A/D converter, a time skew also exists between the channels, resulting in an additional linear phase shift. It is rather surprising to note that this problem does not seem to be recognized in the EEG literature.

In order to illustrate the problem introduced by the phase and gain differences between channels, consider the following two continuous-time signals

$$v_i(t) = \cos(2\pi ft) \quad v_j(t) = A_{ij} \cos(2\pi ft + \phi_{ij}) \quad (2.1)$$

where A_{ij} and ϕ_{ij} is respectively the gain and phase of channel j measured with respect to channel i . Note that the channels are matched when $A_{ij} = 1$ and $\phi_{ij} = 0$. Assume that the same input signal is applied to both channels and that the phase and gain differences are due to the data acquisition system. One measure of the distortion introduced by the recording equipment can be defined as follows

$$NRV(A_{ij}, \phi_{ij}) = \frac{E\{(v_i(t) - v_j(t))^2\}}{E\{v_i^2(t)\}} \quad (2.2)$$

and will be referred to as the normalized residual variance. The above expression can be evaluated by using (2.1), resulting in

$$NRV(A_{ij}, \phi_{ij}) = 2A_{ij}(1 - \cos \phi_{ij}) + (1 - A_{ij})^2 \quad (2.3)$$

It will be shown that the phase differences are more significant than the gain differences in the above equation.

In order to estimate the phase and gain differences between channels, the input channels to the Neuroscience preamplifiers were tied together, sinusoidal frequencies of 5, 10, 20 and 30 Hz were applied separately and the calibration data recorded. A least squares procedure was then used to estimate of the gain and phase of channel i by minimizing the following equation

$$\begin{aligned} SSE_i &= \sum_{n=0}^{N-1} (v_i(n) - B_i \cos(2\pi fnT - \phi_i))^2 \\ &= \sum_{n=0}^{N-1} (v_i(n) - B_{iR} \cos(2\pi fnT) - B_{iJ} \sin(2\pi fnT))^2 \end{aligned} \quad (2.4)$$

with respect to B_{iR} and B_{iJ} , where $B_i = (B_{iR}^2 + B_{iJ}^2)^{1/2}$, $\phi_i = \tan^{-1}(B_{iJ}/B_{iR})$, f is the frequency in Hz, and T is the sampling interval. Next, the average gain of the channels at each frequency was normalized to one while the phase of each channel was referenced to the phase of channel one.

It was found that the phase differences between channel one and the other channels deviated approximately linearly with frequency. In order to correct for the phase differences between channels, a 16 tap FIR filter was designed using the frequency sampling method [25]. The filter was specified to have an linear phase up to approximately 50 Hz, with a high frequency magnitude rolloff, as follows.

f (Hz)	0	7.5	15	22.5	30	37.5	45	52.5	60
Gain	1	1	1	1	1	1	1	0.5	0
Phase	0	ϕ_0	$2\phi_0$	$3\phi_0$	$4\phi_0$	$5\phi_0$	$6\phi_0$	$3.5\phi_0$	0

Table 2.1 - Magnitude and phase characteristics of frequency sampling FIR correction filter. Filter is specified to have a linear phase response up to approximately 50 Hz, with a high frequency magnitude rolloff.

From the measured phase responses, a phase correction factor (ϕ_0) was determined for each channel. Each FIR filter's coefficients was determined by calculating the inverse DFT of the frequency domain filter specifications given in Table 2.1. The channels with the largest gain differences were also corrected. A typical magnitude and phase response of the FIR correction filter is shown in Figure 2.5. The non-linear phase response at the higher frequencies is not significant, for the signal was bandlimited to frequencies below 40 Hz by the analogue input filters.

Each channel of the calibration data was then magnitude and phase corrected with the appropriate FIR filter in order to test the effectiveness of the compensation. The mean and standard deviation of the channel gains and phases for the original and corrected calibration data are shown in the following table. As it can be observed the phase differences were significantly reduced while the magnitude differences were also slightly reduced.

f (Hz)	5	10	20	30
Gain (Original)	1 ± 0.023	1 ± 0.028	1 ± 0.025	1 ± 0.041
Gain (Corrected)	1 ± 0.012	1 ± 0.019	1 ± 0.021	1 ± 0.039
Phase (Original)	0.0 ± 1.9	0.9 ± 3.1	2.3 ± 5.6	4.8 ± 8.2
Phase (Corrected)	-0.5 ± 0.8	-0.2 ± 0.5	0.04 ± 0.39	1.0 ± 0.8

Table 2.2 - Mean and standard deviation of the magnitude and phase responses for the original and corrected calibration data. The phase measurements were referenced to channel one while the magnitude characteristics at each frequency were normalized to one. The corrected calibration data was obtained by applying the FIR correction filter to each channel of the original data. Note the reduction in the standard deviation of the magnitude and phase measurements for the corrected data.

Referring to Table 2.2, a worst case situation for a channel might have $A_y = 1.05$ and $\phi_y = 10$ degrees. Evaluating (2.3) for a phase difference between two channels of 10 degrees with no gain difference ($A_y = 1.0$) results in a normalized residual variance of 3%. For a gain difference between two channels of 5% ($A_y = 1.05$) with no phase difference, the normalized residual variance is 0.25%. Thus a phase difference of 10 degrees is much more significant than a gain difference of 5%. In other words, for the worst case gain and phase

differences typical in the data acquisition system, the phase differences were more significant than the gain differences. However, these differences were significantly reduced by applying the FIR correction filter to the data.

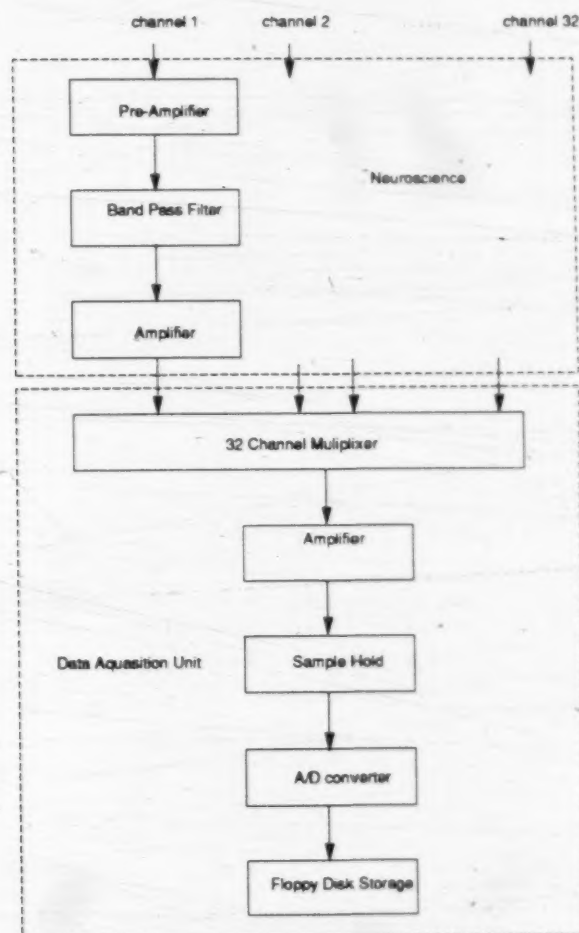


Figure 2.2 - EEG data acquisition system consisting of the Neuroscience Brain Mapping system and a 32-channel data acquisition unit.

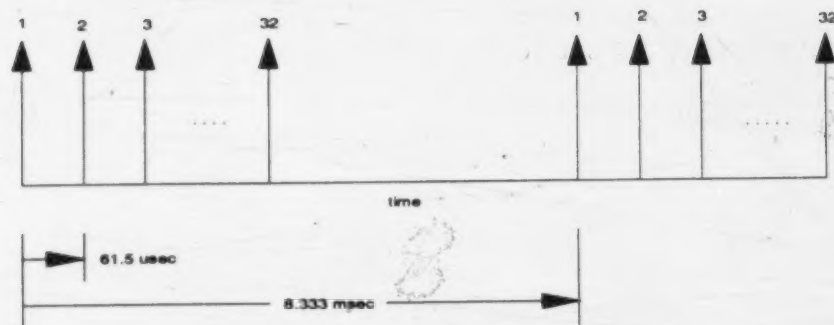


Figure 2.3 - Time skewed sampling of EEG channels. Due to the use of a single A/D converter, each channel is sampled at a difference time, resulting in a linear phase relationship between channels.

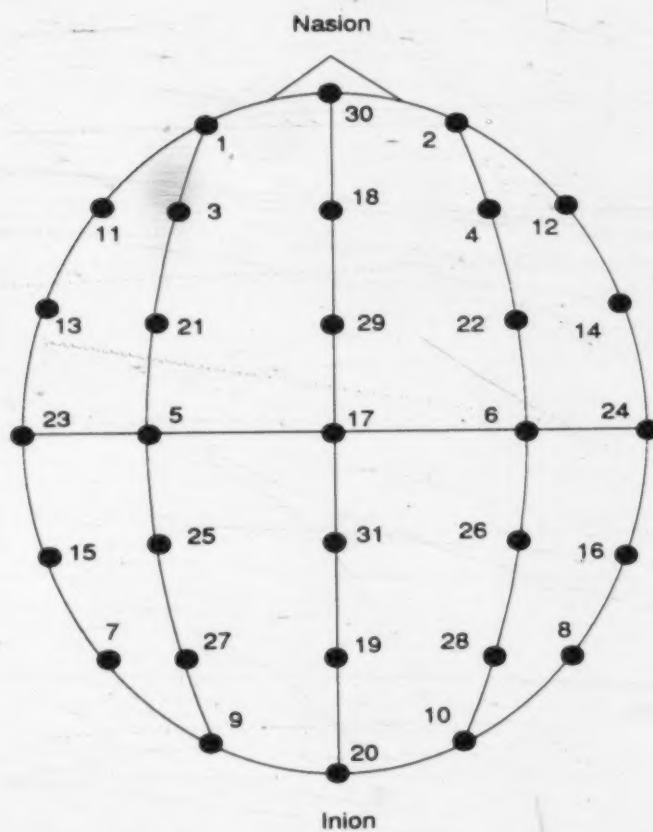


Figure 2.4 - Electrode montage with the reference electrode placed on the left ear lobe.

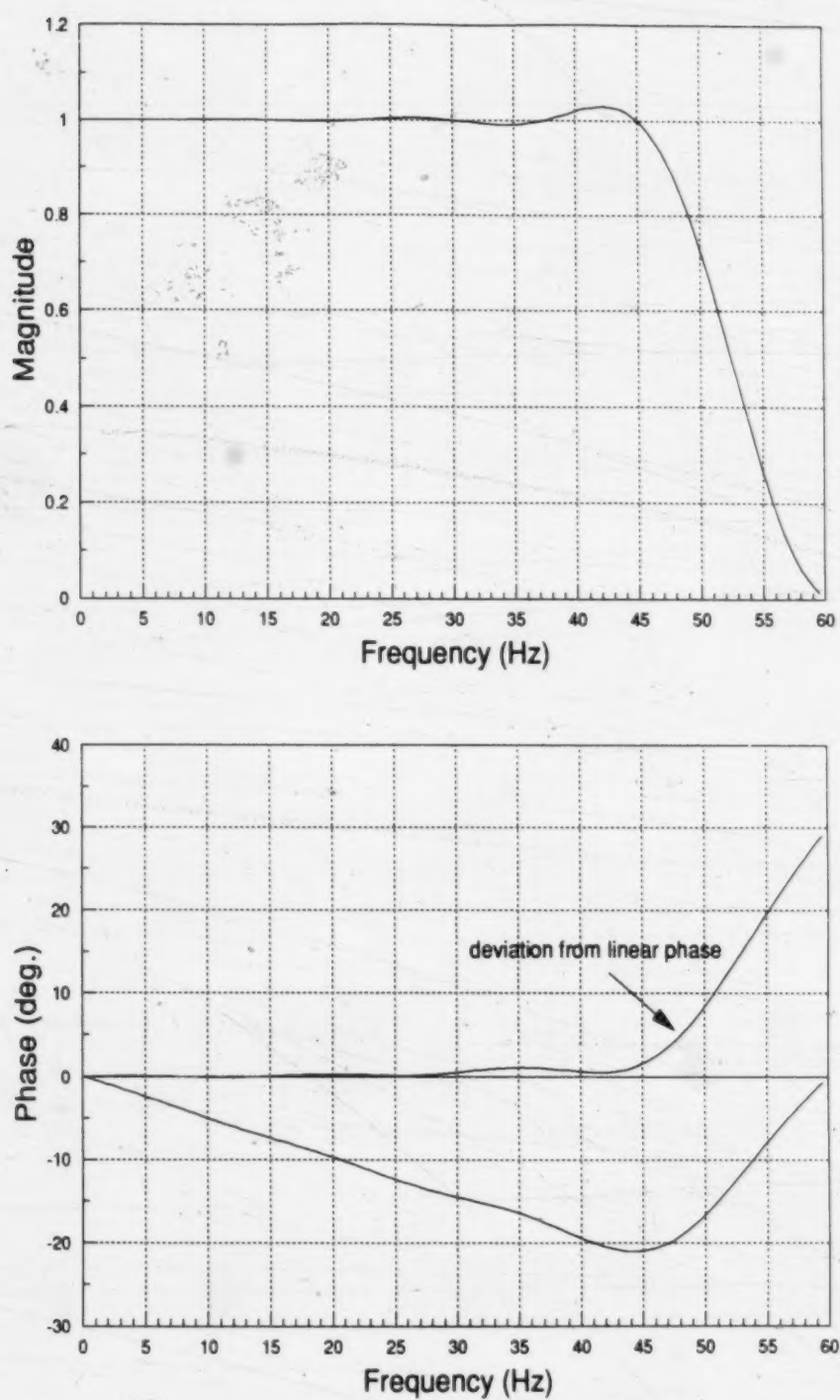


Figure 2.5 - Typical magnitude and phase response of the FIR correction filter. Also shown on the phase response is the deviation from linear phase.

2.4 EEG Data Base and Artifact Editing

A large data base of EEGs was collected over the period of several years at our laboratory by trained personal. Approximately 80 normal and 30 pathologic subjects' EEGs were recorded for two minutes under each the following four conditions.

1. Subject relaxed with eyes closed.
2. Subject relaxed with eyes open and fixated at a single point.
3. Subject performing language skills (non-verbal).
4. Subject performing geometric skills.

Pathologic subjects are defined as those people with known brain disorders, such as tumors or epilepsy. The latter two recording conditions consisted of the subject being presented with visual images that tested either their language or geometric skills.

An analysis of the EEG must acknowledge the presence of significant artifacts [26] [27]. Artifacts are considered as those signals that do not originate from the brain's activity. Sources of artifacts that originate from within the human body include eye and tongue movement, electrocardiograph, muscle activity, and respiration. External artifact sources include electrode noise, 60 Hz electrical noise, amplifier noise and digitizing errors. Since it is very difficult to ensure that the EEG will not contain some artifacts under normal recording conditions, one simply attempts to minimize them.

Since it was found that the language and geometric group EEGs were significantly contaminated by artifacts, only the eyes closed (EC) and eyes positioned (EP) groups were considered for further analysis. The pathologic subjects had a wide variety of brain disorders and drug treatment programs and thus did not constitute a homogeneous group. It is well-known that pharmacological agents can greatly alter the appearance of the EEG. Thus, the two groups considered in the following work were the EC and EP EEGs recorded from normal subjects.

Each of the recorded EEGs were manually edited by trained personal in order to select the 'best' artifact-free 8.53 second (1024 sample) epochs. Each epoch was then digitally processed using the following steps.

1. The temporal average of each channel was removed in order to eliminate an equipment induced offset.
2. The data was digitally low-pass filtered from 0-40 Hz in order to reduce the slight aliasing errors due to the non-ideal analogue antialiasing input filters' response.
3. Each channel was gain and phase corrected, as discussed in this section.

Due to the edge effects of filtering, the first and last 32 points in each epoch were set to zero, resulting in a sequence length of 8.00 seconds (960 points).

From the selection of edited epochs, the best 15 subjects' EEGs recorded under the EC and EP conditions were selected. Epochs that had obvious muscle artifacts or bad channels were rejected. The first three epochs from each selected subject's EEG were used, resulting in 45 epochs for each recording condition. The edited epochs were then subjected to further analysis, as described in later chapters.

It was convenient to define the following symbols to represent the recorded signal in scalar, vector and matrix forms.

$v_k(i)$ - sample i of channel k

$\vec{v}(i) = [v_1(i), v_2(i), \dots, v_{N_s}(i)]^T$ - vector of spatial samples where N_s is the number of spatial samples (normally 31)

$\vec{v}_k = [v_k(1), v_k(2), \dots, v_k(N_t)]^T$ - vector of temporal samples for channel k where N_t is the number of temporal samples (normally 960)

$\vec{V} = [\vec{v}(1), \vec{v}(2), \dots, \vec{v}(N_t)]$ - matrix of size $N_s \times N_t$

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3 Spherical Harmonic Analysis

Harmonic analysis methods have found numerous applications in science and its importance as an analysis tool is unquestioned. The success of harmonic analysis depends upon the suitability of representing the experimental data by a set of harmonic coefficients. It has been found that the solution of many physical problems, ranging from determining the composition of a distant star to the understanding of human speech, can often be simplified by representing the data by a set of harmonic components.

The most commonly used method of performing a harmonic analysis on continuous functions is the Fourier transform. The Fourier transform represents a function by a set of sinusoidal waves and can be used in both single and multi dimensional problems. When the function is sampled on a uniform multidimensional grid, the Fourier transform can be approximated by the discrete Fourier transform (DFT) [1]. The DFT has the advantage that it can be numerically calculated on a digital computer via efficient algorithms. One common application of the DFT is to estimate the power spectral density function of a process [2].

The 2-D DFT has been used to analyze the spatial EEG [3] [4] [5]. The major problem with this approach is that the 2-D DFT is most useful for data that is uniformly rectangularly sampled on the surface of a plane while the EEG is spatially nonuniformly sampled on the curved surface of the scalp. One solution to this problem is to map the electrodes' positions onto a planar surface, then interpolate the data values onto a rectangular grid. Unfortunately, it is generally impossible to map the electrodes' positions onto a plane without distorting the spatial relationships in the data. In addition, the interpolated grid values will differ from the true values. The mapping distortion and the interpolation errors can be made reasonably small by sampling over a limited region of the scalp. However, since the EEG is normally sampled over the top half of the head, the mapping distortion is especially large and the usefulness of the 2-D DFT diminishes. In this situation, the 2-D DFT of the spatial EEG can be viewed as a statistical transformation of the data with no direct relationship to the physics of the problem.

A better approach to analyzing the spatial EEG is to consider the spherical harmonic expansion of the data. The spherical harmonic expansion of a potential function assumes spherical geometry in contrast to the planar geometry of the 2-D DFT. It is obviously more reasonable to approximate the scalp's surface by the surface of a sphere as described in the previous chapter. Although spherical harmonic analysis methods have been employed in several areas, such as in geophysics [6] [7], there are no published results of its application to the spatial EEG.

This chapter considers the problem of determining the spherical harmonic expansion of a function and a sampled function. The first several sections review the theory of the spherical harmonic expansion of a spherical function along with several applications and properties of spherical harmonic expansions. The last part of the chapter considers the problems introduced by having knowledge of the function on only a limited region of the sphere and by having only samples of the function.

3.1 Spherical Harmonic Expansion of a Spherical Function

It is possible to represent an analytic function in terms of a weighted sum of orthogonal basis functions. The selection of the basis functions depends upon the coordinate geometry of the function. For example, a one dimensional function can usually be

represented by sine and cosine functions. Three-dimensional and spherical functions can be represented by spherical harmonic functions. This section reviews the theory related to obtaining the spherical harmonic expansion of a continuous function.

Assume that the value of a function v is known at all points on the surface of a sphere and that the function has a finite number of singularities. Let the function be represented in spherical coordinates as $v = f(\phi, \theta)$ where ϕ, θ are the tip and spin angles respectively, as shown in Figure 3.1. It is desired to represent v by a harmonic expansion.

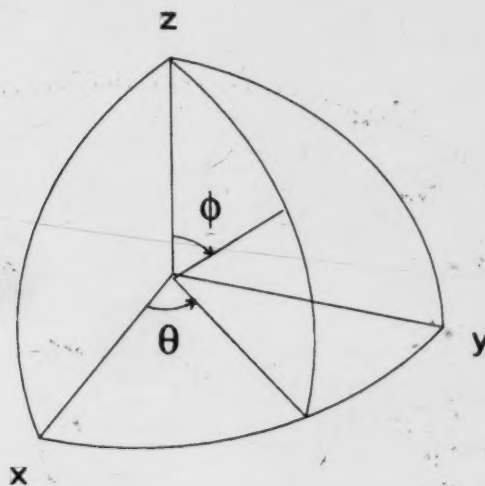


Figure 3.1 - Spherical coordinate system. The relationships between the x-y-z and θ - ϕ coordinates are $x = \cos \theta \sin \phi$, $y = \sin \theta \sin \phi$ and $z = \cos \phi$. It is convenient to assume that the positive z-axis points towards the north pole of the sphere.

The surface spherical harmonic function of integral degree n and order m [8] [9] is defined by

$$\begin{aligned} U_n^m &= \cos(m\theta) P_n^m(\cos \phi) \quad m > 0 \\ &= P_n(\cos \phi) \quad m = 0 \\ &= \sin(-m\theta) P_n^{-m}(\cos \phi) \quad m < 0 \end{aligned} \tag{3.1.1}$$

where P_n is the Legendre polynomial of degree n and P_n^m is the associated Legendre function of the first kind of degree n and order m . Refer to Appendix A and [10] for more information on Legendre polynomials and associated functions. Explicit expressions for several low degree spherical harmonic functions are as follows.

$$U_0^0 = 1$$

$$U_1^0 = \cos \phi \quad \begin{aligned} U_1^1 &= \cos \theta \sin \phi \\ U_1^{-1} &= \sin \theta \sin \phi \end{aligned}$$

$$U_2^0 = \frac{1}{2}(3 \cos^2 \phi - 1) \quad \begin{aligned} U_2^1 &= 3 \cos \theta \sin \phi \cos \phi & U_2^2 &= 3 \cos(2\theta) \sin^2 \phi \\ U_2^{-1} &= 3 \sin \theta \sin \phi \cos \phi & U_2^{-2} &= 3 \sin(2\theta) \sin^2 \phi \end{aligned}$$

It can be observed that the spherical harmonic functions are the product of two one-dimensional functions. Along the longitudinal paths on the sphere, which intersect the poles, the functions are defined by the Legendre polynomials and associated functions. A plot of several of the Legendre polynomials and the associated functions is shown in Figure 3.2. Along the latitudinal paths on the sphere, the spherical harmonic functions vary sinusoidally.

The spherical harmonic functions can be classified in terms of zonal, sectorial and tesseral harmonics, depending upon the values of m and n , as follows. Also, refer to Figure 3.3 for a plot of the paths along where the functions can be zero.

- (a) Zonal Harmonics ($m = 0$). The function U_n^0 is independent of θ and thus has a fixed value along a path of constant latitude. Since the function $P_n(\cos \phi)$ has n distinct zeros in the range $0 \leq \phi \leq \pi$, the zeros of the zonal harmonic function form n parallel circles. The point on the sphere defined by $\phi = 0$ is referred to as the pole of the zonal harmonic.
- (b) Sectorial Harmonics ($m = n$). It can be shown that U_n^n is independent of ϕ and thus does not vary along longitudinal paths. The zeros of the function form n distinct great circles that all pass through the pole $\phi = 0$, with the angle between adjacent planes of the great circles equalling π/n .
- (c) Tesseral Harmonics ($0 < |m| < n$). The function has $|m|$ sectorial zeros and $n - |m|$ zonal zeros which intersect orthogonally along with two zero points at $\phi = 0$ and $\phi = \pi$. Most of the spherical harmonics are of this type.

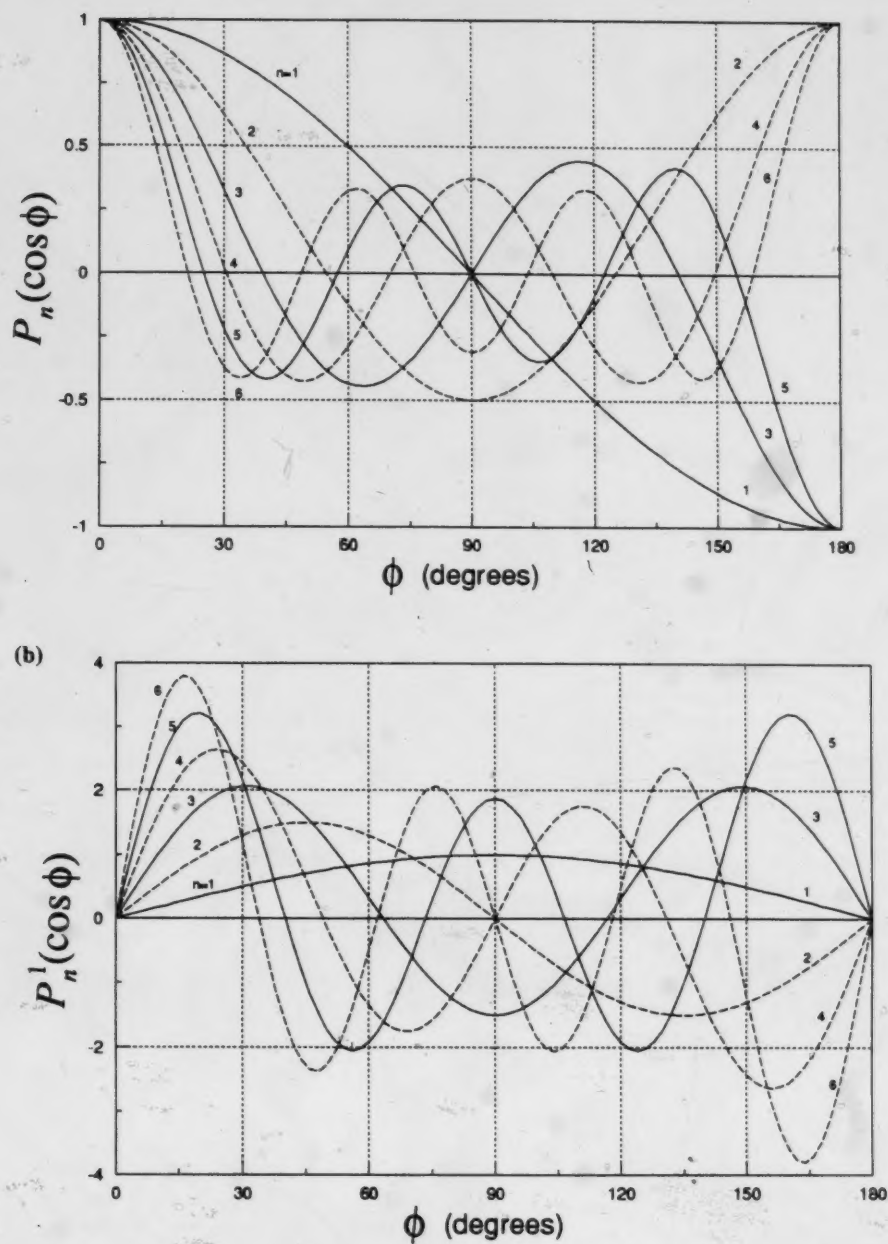


Figure 3.2 - Plot of $P_n(\cos \phi)$ and $P'_n(\cos \phi)$ vs. ϕ .

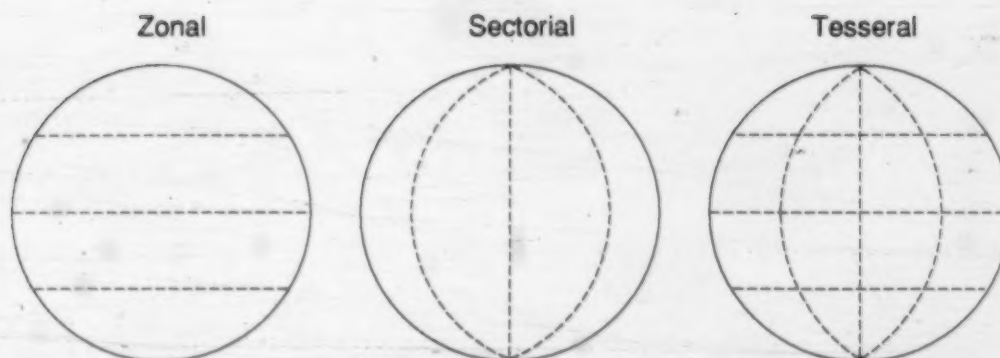


Figure 3.3 - Typical plot of the zero value paths for the zonal, sectorial and tesseral harmonics can have zero value. The zonal and sectorial harmonics have a constant value along latitudinal and longitudinal paths respectively. Tesseral harmonics have only a constant value of zero along some longitudinal and latitudinal paths.

It is convenient to define a set of normalized spherical harmonic functions as follows.

$$Y_n^m = 2\sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}} U_n^m \quad m \neq 0 \quad (3.1.2)$$

$$= \sqrt{\frac{2n+1}{4\pi}} U_n^0 \quad m = 0$$

This choice of normalization results in Y_n^m having a squared surface integral of unity, that is

$$\int_S (Y_n^m)^2 ds = 1 \quad (3.1.3)$$

where the integration is performed over the surface of the sphere. The spherical harmonic expansion of v in terms of normalized spherical harmonics is given by

$$v = \sum_{n=0}^{\infty} \sum_{m=-n}^n a_n^m Y_n^m \quad (3.1.4)$$

where a_n^m is the spherical harmonic coefficient of order m and degree n . Thus, the spherical harmonic expansion of v involves the determination of an infinite set of spherical harmonic coefficients.

The normalized spherical harmonic functions form an orthonormal set of functions, that is

$$\int_s Y_n^m Y_q^p ds = 1 \quad (m = p, n = q) \quad (3.1.5)$$

$$= 0 \quad \text{otherwise}$$

An important consequence of representing a function in terms of a weighted sum of orthogonal functions is that the weighting coefficients can be determined from an uncoupled set of equations. If for some reason an error was made in the solution for one of the coefficients, this will not affect the solution for the remaining coefficients.

The solution for the spherical harmonic coefficients is greatly simplified by the orthogonality of the spherical harmonic functions and is given by

$$a_n^m = \int_s v Y_n^m ds \quad (3.1.6)$$

The relationships between the function v and its spherical harmonic expansion, given by equations (3.1.4) and (3.1.6), can be considered as a transform pair. It is important to note that the transforms are invertible, and that the representation of a function by a spherical harmonic expansion completely defines that function. The following section illustrates some of the advantages of representing a function in terms of its spherical harmonic expansion.

3.2 Properties and Applications of Spherical Harmonic Expansions

3.2.1 Multipole Expansion Equivalence

A spherical harmonic expansion of a potential function is equivalent to a multipole expansion about the origin of the coordinate system. Each spherical harmonic function relates to a specific multipole degree and orientation, as will be explained next.

The function Y_0^0 , which represents a spatially constant value on the surface of a sphere, is equivalent to the potential function generated by a monopole charge located at the origin of the coordinate system while the a_0^0 coefficient relates its strength. Also, the magnitude of the potential function on a spherical surface due to the monopole is inversely proportional to the radius of the sphere.

The 3 spherical harmonic functions of degree one relate to the potential function of a degree 1 multipole (dipole) located at the center of the sphere. The relationship between the dipole's component strengths (D_x, D_y, D_z) and the three degree one coefficients are as follows.

$$D_x = a_1^1 \quad D_y = -a_1^{-1} \quad D_z = a_1^0 \quad (3.2.1)$$

Thus, given the three degree one coefficients, it is possible to determine the strength and orientation of a centrally located dipole. In addition, the potential function of a dipole is inversely proportional to the square of the distance from the origin.

The 5 spherical harmonic functions of degree two relate to the potential function of a degree 2 multipole (quadrupole) located at the center of the sphere. In addition, the potential function is inversely proportional to the cube of the radius of the sphere. Continuing this expansion equivalence for higher degrees, it can be shown that a linear combination of the $2n + 1$ spherical harmonic functions of degree n can be used to represent the potential function of a degree n multipole located at the center of a sphere.

3.2.2 Rotation Invariance

An important property of a spherical harmonic expansion of a function can be inferred from the multipole equivalence discussed in the previous section. Assume that a degree n multipole is located at the center of the sphere and is represented by a linear combination of the degree n spherical harmonic functions. The original coordinate system is now rotated about the origin (center of the sphere) and the new spherical harmonic expansion is determined about the new coordinate system. It is easy to infer that the new spherical harmonic expansion must still only involve the degree n spherical harmonic coefficients since a rotation of the coordinate system cannot change the degree of the multipole.

The rotation invariance of the degree n spherical harmonic coefficients can now be stated. Assume that a spherical harmonic expansion of a potential function is determined about a given coordinate system. A rotation of the coordinate system changes the spherical harmonic coefficients of degree n into a new set of coefficients of the same degree.

3.2.3 Relation to Potential Theory

The extension of the surface spherical harmonic into a three-dimensional harmonic function V_n^m is accomplished via

$$V_n^m = r^n U_n^m \quad (3.2.2)$$

where r is the radius of the spherical surface. The function V_n^m is referred to as a solid spherical harmonic since it is a three-dimensional solution to Laplace's equation

$$\nabla^2 V_n^m \equiv \frac{\partial^2 V_n^m}{\partial x^2} + \frac{\partial^2 V_n^m}{\partial y^2} + \frac{\partial^2 V_n^m}{\partial z^2} = 0 \quad (3.2.3)$$

Evaluating Laplace's equation in spherical coordinates results in

$$\nabla^2 V_n^m = r \frac{\partial^2 r V_n^m}{\partial r^2} + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial V_n^m}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 V_n^m}{\partial \theta^2} = 0 \quad (3.2.4)$$

which can be simplified using (3.2.2) into

$$\nabla^2 V_n^m = n(n+1)U_n^m + \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial U_n^m}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 U_n^m}{\partial \theta^2} = 0 \quad (3.2.5)$$

Due to the importance of Laplace's equation in potential theory and the fact that the solid spherical harmonics satisfy that equation, spherical harmonic functions often appear in the solutions to problems in potential theory.

One useful property of a spherical harmonic expansion of a potential function defined on one spherical surface is the ability to determine the potential function on another surface. Define three concentric spherical shells, located in a homogeneous medium, of radii (r_o, r, R) where $(R, r) > r_o$. Assume r_o is the smallest radius of a spherical region that contains a set of charges. Let the function v describe the potential function on a spherical surface of radius R due only to a set of charges contained inside the sphere and let it be represented by

$$v = \sum_{n=0}^{\infty} Y_n \quad (3.2.6)$$

where

$$Y_n = \sum_{m=-n}^n a_n^m Y_n^m$$

The problem definition is: given value of the potential function v on a spherical surface of radius R , determine the potential function on a spherical surface of radius r .

The solution to this problem can again be inferred from the multipole expansion equivalence of a spherical harmonic expansion. It was inferred that the potential function on a spherical surface of radius r due to a degree n multipole located at the origin is inversely proportional to $r^{(n+1)}$. By considering this relationship for the two spherical surfaces, the potential function on the surface of a sphere of radius r is then given by

$$v(r) = \sum_{n=0}^{\infty} \left(\frac{r}{R} \right)^{n+1} Y_n \quad (3.2.7)$$

A similar equation can also be derived for situations where there are internal and external charges.

3.2.4 Spatial Laplacian

The spatial Laplacian has a special significance in regard to the EEG. It has been shown [11] that the spatial Laplacian of the potential function measured on the scalp gives an estimate of the volume current flowing radially to the scalp. The spatial Laplacian is not affected by current flowing tangentially in the scalp and is independent of the activity of

the reference electrode (effect of the reference electrode is discussed in a following chapter). The spatial Laplacian, which acts as a high pass spatial filter, is used to localization the activity of the EEG.

The surface Laplacian of a two dimensional function v defined in spherical coordinates is given by

$$\nabla^2 v = \frac{1}{\sin \phi} \frac{\partial}{\partial \phi} \left(\sin \phi \frac{\partial v}{\partial \phi} \right) + \frac{1}{\sin^2 \phi} \frac{\partial^2 v}{\partial \theta^2} \quad (3.2.8)$$

Using the results of (3.2.5), the above equation can be expressed as

$$\nabla^2 v = - \sum_{n=0}^{\infty} n(n+1) Y_n \quad (3.2.9)$$

The above summation converges to a finite value as long as v is analytic on the surface of the sphere.

3.2.5 Spatial Power Spectrum

Another application of a spherical harmonic expansion is to estimate the spatial power spectrum [12], defined as

$$S_n = \sum_{m=-n}^n (a_n^m)^2 \quad \text{for } 0 \leq n \leq \infty \quad (3.2.10)$$

It has been shown [13] that S_n is invariant with respect to rotations of the coordinate system, although the spherical harmonic coefficients are rotation variant, as previously discussed. The spatial power spectrum represents the distribution of energy in v vs. wavelength, with increasing values of n relating to decreasing wavelengths. The energy in the original function can be shown to be related to the spatial power spectrum by

$$\int_S v^2 ds = \sum_{n=0}^{\infty} S_n \quad (3.2.11)$$

The above equation can be considered as the analog of Parseval's theorem for spherical functions and is the result of the orthogonality of the spherical harmonic functions.

It should be noted that there is not a direct relationship between the degree of a spherical harmonic function and a wavelength. For example, consider the zero order spherical harmonic functions, which only involve the Legendre polynomials. The Legendre polynomials can be written as a linear combination of sinusoidal functions and vice versa as follows.

$$\begin{aligned}
P_1(\cos \phi) &= \cos \phi & \cos \phi &= P_1(\cos \phi) \\
P_2(\cos \phi) &= \frac{1}{4}(3 \cos 2\phi + 1) & \cos 2\phi &= \frac{1}{3}(4P_2(\cos \phi) - P_0(\cos \phi)) \\
P_3(\cos \phi) &= \frac{1}{8}(5 \cos 3\phi + 3 \cos \phi) & \cos 3\phi &= \frac{1}{5}(8P_3(\cos \phi) - 3P_1(\cos \phi))
\end{aligned}$$

Considering a sinusoidal expansion of a Legendre polynomial of degree n , the sinusoidal term with the shortest wavelength is $\cos(n\phi)$. For a sphere of radius R , this represents a wavelength of $2\pi R/n$ along the longitudinal paths and is the most convenient definition of the wavelength of a degree n spherical harmonic.

Using the above definition for longitudinal wavelength, the spherical harmonic Y_n^m has a wavelength of $2\pi R/(n-m)$. The latitudinal wavelength of Y_n^m depends upon where on the sphere it is measured. At the spheres equator, the latitudinal wavelength is $2\pi R/m$.

3.3 Spherical Harmonic Expansion of an Incomplete Function

This section considers the problem of determining the spherical harmonic expansion of a function that is known only over a limited region of the surface of a sphere. The motivation for investigating this problem is that it will be shown in a following chapter that the EEG is sampled over approximately the surface of a hemisphere. As a result of the hemispherical sampling, most of the spherical harmonic functions become non-orthogonal on the surface of a hemisphere and a least squares or similar approach is required to determine the spherical harmonic coefficients.

In order to illustrate the problem introduced by sampling a function over a limited region of the sphere, consider the surface integral of the product of two spherical harmonics of order m and p and degrees n and q respectively.

$$R(m, n; p, q; \phi_0) = \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} Y_n^m Y_q^p \sin(\phi) d\phi d\theta \quad (3.3.1)$$

Equation (3.3.1) can be considered as a measure of the non-orthogonality (correlation) between Y_n^m and Y_q^p over a spherical cap of size defined by ϕ_0 . When $R(m, n; p, q; \phi_0)$ equals zero, Y_n^m and Y_q^p are orthogonal over the surface of integration. A non-zero value for $R(m, n; p, q; \phi_0)$ indicates that the functions are not orthogonal, with the magnitude indicating the amount of non-orthogonality.

Considering only the zero order functions, (3.3.1) can be simplified to

$$R(0, n; 0, q; \phi_0) = 2\pi \int_{\phi=0}^{\phi_0} Y_n^0 Y_q^0 \sin(\phi) d\phi \quad (3.3.2)$$

A plot of the above equation for $n=0$ and $n=1$ as a function of ϕ_0 and q is shown in Figure 3.4. As expected, the functions Y_0^0 and Y_1^0 become orthogonal to Y_q^0 when the value of the

function is known over the entire surface of the sphere ($\phi_0 = \pi$). Equation (3.3.2) was evaluated for several combinations of zero order harmonic functions and for $\phi_0 = \pi/2$, that is, when the value of the functions is known only on the surface of a hemisphere, with the results shown in Table 3.1. It can be noted that approximately half of the results are zero while the largest non-zero values occurs when $q = n + 1$, excluding the $n = q$ terms.

n q	0	1	2	3
0	0.500			
1	0.433	0.500		
2	0	0.242	0.500	
3	-0.166	0	0.370	0.500
4	0	-0.054	0	0.279
5	0.104	0	-0.145	0
6	0	0.024	0	-0.075

Table 3.1 - Table of $R(0, n; 0, q; \pi/2)$ vs n and q . These results show a measure of the orthogonality between Y_n^0 and Y_q^0 when the function is known only on the surface of a hemisphere. A zero value in the table indicates that the two functions are orthogonal while the maximum value of 0.5 results from the fact that only half the sphere is used.

Since the spherical harmonic functions are generally not orthogonal when they are known only on the surface of a hemisphere, it is not possible to solve for the spherical harmonic coefficients by inverting (3.1.4) into a closed form similar to (3.1.6). Instead, one approach to determining the spherical harmonic coefficients is to minimize the following equation, which represents the integrated squared difference between the known function and its spherical harmonic expansion representation

$$\int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} \left(v - \sum_{n=0}^{\infty} \sum_{m=-n}^n a_n^m Y_n^m \right)^2 \sin(\phi) d\phi d\theta \quad (3.3.3)$$

with respect to the spherical harmonic coefficients. Taking the partial derivative of the above expression with respect to a_n^m and equating the result to zero gives, after some rearrangement (refer to appendix B for details),

$$\sum_{q=0}^{\infty} \sum_{p=-q}^q a_q^p R(m, n; p, q; \phi_0) = \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} v Y_n^m \sin(\phi) d\phi d\theta \quad (3.3.4)$$

where $0 \leq n \leq \infty$ and $-n \leq m \leq n$. The above equation can be further simplified by noting that

$$R(0, n; p, q; \phi_0) = 0 \quad (p \neq 0) \quad (3.3.5a)$$

$$R(m, n; p, q; \phi_0) = 0 \quad (m \neq n) \vee (p \neq q) \quad (3.3.5b)$$

Result (3.3.5a) states that the zero order spherical harmonic functions are orthogonal to the non-zero order functions, while result (3.3.5b) infers that the non-zero order spherical harmonic functions remain orthogonal to all of the spherical harmonic functions. Thus the equations for the zero order coefficients are independent of the equations for the non-zero order coefficients, and more importantly, the equations for the non-zero order coefficients are uncoupled. Using these results, (3.3.4) can be simplified to

$$\sum_{q=0}^{\infty} a_q^0 R(0, n; 0, q; \phi_0) = \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} v Y_n^0 \sin(\phi) d\phi d\theta \quad (3.3.6a)$$

$$a_n^m = \frac{1}{R(m, n; m, n; \phi_0)} \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} v Y_n^m \sin(\phi) d\phi d\theta \quad \text{for } (m \neq 0) \quad (3.3.6b)$$

It can be observed from the above equations that the major effect of knowing the value of a function only over the surface of a spherical cap is the coupling of the equations relating to the solution for the zero order spherical harmonic coefficients. The solution for the non-zero order coefficients is basically unaffected.

Given the value of the function v over the surface of a spherical cap, equation (3.3.6a) defines an infinite set of coupled linear equations in an infinite number of unknowns, namely the zero order spherical harmonic coefficients. In order to derive an explicit expression for the zero order spherical harmonic coefficients, one could truncate the expansion to a finite degree N . However, this truncation will generally introduce an error in the estimates for the zero order coefficients, with the largest errors occurring for those coefficients with degrees nearest N . By truncating the expansion at a suitably high degree, good estimates can be obtained for the low degree zero order coefficients. Fortunately, most functions can be adequately approximated by using only the low degree coefficients. Truncation of the expansion does not affect the estimates obtained for the non zero order coefficients.

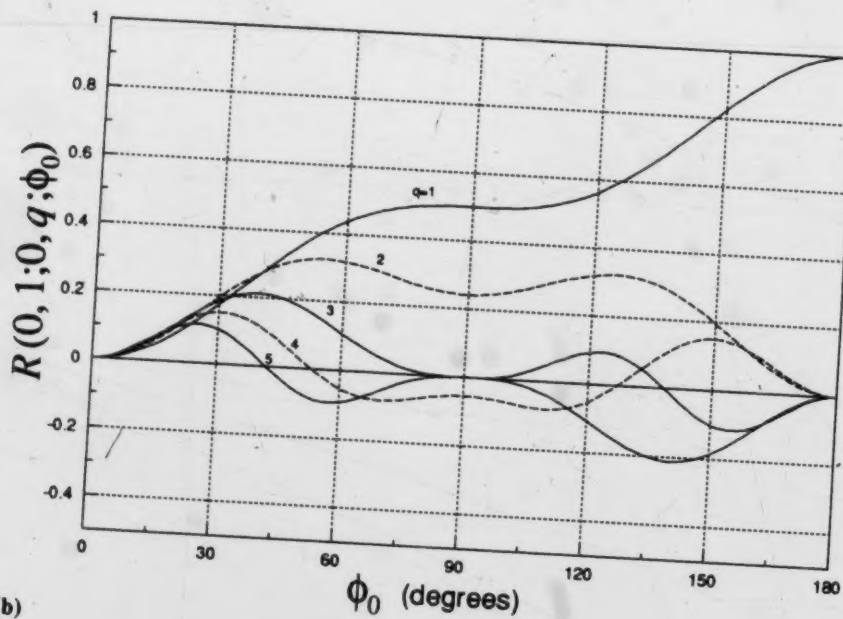
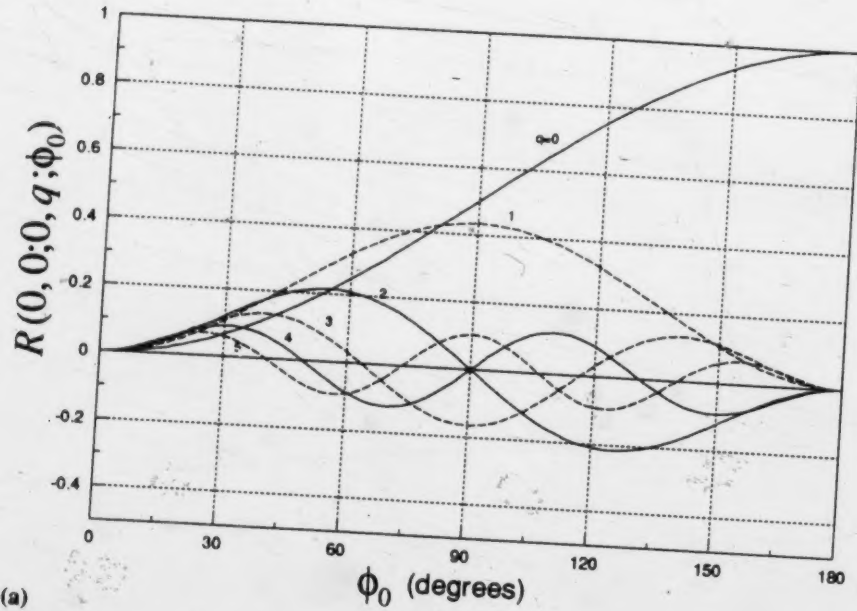


Figure 3.4 - Plot of $R(0, n; 0, q; \phi_0)$ vs. ϕ_0 for (a) $n=0$ and (b) $n=1$ and q ranging from 0 to 5. Figure (a) plots a measure of the non orthogonality between Y_0^0 and Y_1^0 while Figure (b) shows the same measure between Y_1^0 and Y_2^0 when the functions are known only on the surface of a spherical cap of size defined by ϕ_0 .

3.4 Spherical Harmonic Expansion of a Sampled Function

Several problems are introduced in determining the spherical harmonic expansion of a sampled function. Sampling a function introduces the problem of aliasing. A related problem is that all of the spherical harmonic coefficients cannot be determined from a finite number of samples, thus requiring truncation of the spherical harmonic expansion to a finite degree. Another problem introduced by sampling is that the sampled spherical harmonic functions are not, in general, orthogonal even if the functions are sampled approximately uniformly over the complete surface of the sphere. The final problem, related specifically to the EEG, is when the potential distribution is sampled over approximately the surface of a hemisphere which further increases the non-orthogonality of the sampled spherical harmonic functions.

A fundamental aspect that must be considered when performing a harmonic analysis is determining the adequacy of the sampling. Sampling theory states that the sampling rate must be at least twice the highest frequency component in the signal in order to prevent aliasing. Temporal data acquisition systems use an analogue low pass filter to band-limit the signal. For spatially sampled signals, the physical size of the sensor can be used to spatially bandlimit the signal. A typical electrode used to sample the EEG contacts the skull in a circular region with a diameter of about 1/2 cm., which results in attenuation of signals whose wavelength is smaller than 1/2 cm. However, the 31-electrode montage used to acquire the EEG has an electrode spacing of approximately 8 cm and sampling theory states that the shortest wavelength should be 16 cm. It is therefore obvious that the EEG electrodes' do not adequately spatially bandlimit the sampled signal. Fortunately, the spatial EEG appears to be dominated by longer spatial wavelengths and thus the effects of aliasing are minimal, as will be shown in a following chapter.

Due to the finite number of samples, the spherical harmonic expansion of a sampled function should be truncated to a finite summation as

$$\bar{v} = \sum_{n=0}^N \sum_{m=-n}^n a_n^m Y_n^m \quad (3.4.1)$$

where $\bar{v}$ is an approximation to v and N is the degree of truncation. One can easily show that the number of spherical harmonic coefficients in (3.4.1) equals $(N+1)^2$. Assume that N_s spatial samples of the function v are known. Let the spatial samples be combined into a N_s sized vector $\bar{v}$ and assume that the spherical coordinates of the samples are known. The sampled version of (3.4.1) can be written in matrix form as follows

$$\bar{v} = \bar{Y} \bar{a} \quad (3.4.2)$$

where

$$\bar{a} = [a_0^0 \quad a_1^0 \quad a_1^1 \quad a_1^{-1} \dots a_N^{-N}]^T$$

$$\bar{Y} = \begin{bmatrix} Y_0^0 & Y_1^0 & Y_1^1 & Y_1^{-1} & \dots & Y_N^{-N} \\ Y_0^0 & Y_1^0 & Y_1^1 & Y_1^{-1} & \dots & Y_N^{-N} \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \end{bmatrix}$$

$\bar{v}$ is a N_s sized vector, $\bar{Y}$ is a N_s by $(N+1)^2$ sized matrix and where the i -th row in $\bar{Y}$ is evaluated using the spherical coordinate of the i -th spatial sample. The main problem with the sampled data is that the columns of $\bar{Y}$ are not orthogonal to each other. In other words,

the sampled spherical harmonic functions are not orthogonal, although the continuous versions are and a discrete version of (3.1.6) cannot be used to solve for the spherical harmonic coefficients.

One method of solving for the spherical harmonic coefficients is to use a least squares approach. The least squares estimate of the spherical harmonic coefficients minimizes the sum of the squared differences between the elements in $\vec{v}$ and $\vec{Y}\vec{a}$. Mathematically, $||\vec{v}(k) - \vec{Y}\vec{a}||$ is minimized with respect to $\vec{a}$, resulting in

$$\vec{a} = (\vec{Y}^T \vec{Y})^{-1} \vec{Y}^T \vec{v} \quad (3.4.3)$$

with the assumption that the system of equations is not singular. Although it is possible to use (3.4.3) to solve for the spherical harmonic coefficients, the set of equations is often ill-conditioned. A more numerically stable solution can be obtained with the use of the singular value decomposition.

Singular value decomposition (SVD) has been shown to be intrinsically related to the solution of least squares problems [14]. It is well-known that the SVD algorithm is numerically stable and provides reliable solutions to even ill-conditioned problems. Another use of the SVD is to estimate the rank of a matrix and to reduce the effects of noise. Assuming that $N_s \geq (N+1)^2$, the SVD of $\vec{Y}$ is given by

$$\vec{Y} = \vec{A} \vec{\Sigma} \vec{C}^T \quad (3.4.4)$$

where $\vec{A}$ and $\vec{C}$ are respectively N_s by $(N+1)^2$ and $(N+1)^2$ by $(N+1)^2$ sized orthonormal matrices and where $\vec{\Sigma}$ is a diagonal matrix. The columns of $\vec{A}$ and $\vec{C}$ are the eigenvectors of $\vec{Y}\vec{Y}^T$ and $\vec{Y}^T \vec{Y}$ respectively while the diagonal elements in $\vec{\Sigma}$ are the square roots of the non-zero eigenvalues of $\vec{Y}\vec{Y}^T$. The diagonal elements in $\vec{\Sigma}$ are referred to as the singular values of $\vec{Y}$ and are arranged in decreasing magnitude. By using the SVD factorization of $\vec{Y}$, (3.4.3) can be written as

$$\vec{a} = \vec{C} \vec{\Sigma}^{-1} \vec{A}^T \vec{v} \quad (3.4.5)$$

Although equations (3.4.3) and (3.4.5) are mathematically equivalent, the latter method is numerically more stable.

Sampling the function v at many points approximately uniformly distributed over the surface of the sphere will result in the columns of $\vec{Y}$ being approximately orthogonal and of full rank, assuming that N_s exceeds the number of coefficients $(N+1)^2$. As N_s is reduced while N is held constant, the columns of $\vec{Y}$ will become more non-orthogonal and eventually $\vec{Y}$ will become singular. A singular matrix indicates that at least one column in $\vec{Y}$ is a linear combination of two or more other columns. When the columns in $\vec{Y}$ involve sampled harmonic functions, a singular matrix infers that at least one shorter wavelength appears as a linear combination of two or more longer wavelengths, which is a form of aliasing.

The occurrence of aliasing can be easily determined with the aid of the singular values of $\bar{Y}$. It can be shown that the rank of $\bar{Y}$ is the number of non-zero singular values. Assuming that the $(N+1)^2 < N_s$ and that $\bar{Y}$ does not contain aliased components, the rank of $\bar{Y}$ should be $(N+1)^2$. Thus, aliasing has occurred if the number of non-zero singular values is less than $(N+1)^2$.

The problem of fitting a truncated spherical harmonic expansion to the sampled data is actually a subset of a much larger class of linear modeling problems. Modeling problems are usually classified into one of three general classes depending upon the relationship between the number of constraints (N_s) and the number of unknowns (spherical harmonic coefficients) as follows.

1. Overdetermined models $(N+1)^2 < N_s$. Problems of this class have fewer unknowns than data points, thus the fitted function does not have enough degrees of freedom to match the given information. The least-squares approach, as discussed in this section, is commonly used to solve for the model parameters and results in the estimated function representing a smoothed version of the actual function.
2. Exactly determined models $(N+1)^2 = N_s$. This class represents the ideal situation where the number of degrees of freedom in the model and in the data match. Although it is possible to invert (3.4.2) directly without resorting to a least-squares approach, the solution is highly sensitive to the effects of noise.
3. Underdetermined models $(N+1)^2 > N_s$. The model has more degrees of freedom than constraints, thus an infinite number of solutions match the constraints. In order to arrive at a unique solution, additional constraints must be imposed. One such constraint to be considered in a following chapter is to minimize the bending energy of the estimated function. The actual form of the constraint usually is determined from knowledge about the physical process that generated the signal.

The SVD can also be used to solve underdetermined problems. In order to arrive at a unique solution, the SVD matches the model to the given data along with minimizing the norm of the coefficient vector, as follows.

$$\min ||\bar{a}|| \quad \text{subject to} \quad ||\bar{v} - \bar{Y}\bar{a}|| = 0 \quad (3.4.6)$$

The solution to this problem is still given by (3.4.5) if only the non-zero diagonal elements in $\bar{\Sigma}$ are inverted. Due to the choice of normalization of the spherical harmonic functions, this solution can be considered as the minimum energy fit. That is, it also solves the following dual problem

$$\min \int_s v^2 ds \quad \text{subject to} \quad ||\bar{v} - \bar{Y}\bar{a}|| = 0 \quad (3.4.7)$$

The usefulness of this solution depends upon how appropriate the constraint of minimum energy is to the properties of the actual signal.

As a final note, it can be shown that $\bar{v}$ can be written as

$$\bar{v} = \bar{S}v \quad (3.4.8)$$

where $\bar{S} = \bar{Y}(\bar{Y}^T \bar{Y})^{-1} \bar{Y}^T$. The N , by N , sized matrix $\bar{S}$ is often referred to as the similarity matrix, for it defines the mapping of the observed values into the estimated values. It can be shown that if the model is underdetermined, then $\bar{S}$ will equal the identity matrix and the data estimate will be the same as the observed values. For overdetermined models, all of the elements in $\bar{S}$ will generally be non-zero. The direct evaluation of the above equation for $\bar{S}$ is usually not recommended due to the effects of numerical rounding. A more stable solution for $\bar{S}$ is again provided by using the SVD of $\bar{Y}$, and can be written as

$$\bar{S} = \bar{A} \bar{A}^T \quad (3.4.9)$$

The concept of the similarity matrix will be used in the following chapters.

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4 Reference Problem

It is well-known that the interpretation of electric potential measurements obtained on the scalp is complicated by the activity of the reference electrode [1]. Potential measurements indicate the difference in the potentials between two points, and generally not the potential at a specific point. Only when one of the measurement points is known to be electrically neutral can the recorded potential be attributed to activity at the other point.

The EEG is usually recorded with respect to a common reference electrode when the EEG will be subjected to a quantitative spatial analysis. Unfortunately, there is no electrically neutral point on the human body to place the reference electrode. Therefore, the measured activity must only be considered as the potential difference between each electrode and the reference electrode. As outlined by numerous investigators, the measured activity can be modified by simply moving the reference electrode to a different location. Thus, temporal analysis of the EEG recorded with respect to a common reference should consider the effects of the active reference on the analysis method in order to avoid misleading and possibly incorrect results.

In order to avoid the reference problem associated with a common reference recording, the clinical EEG is usually bipolarly recorded. Bipolar recordings are obtained by measuring the potential difference between two closely spaced electrodes. Multiple sets of bipolar electrodes are used to sample the differential potential distribution on the scalp. Since bipolar recordings do not have a common reference electrode, the problem of determining a suitable location for the reference electrode is avoided. However, the response of a bipolar electrode pair is directional since rotations of the electrode pair will generally record different signals. Also, the potential function cannot generally be recovered given only differential measurements. Thus, it can be concluded that the bipolar recording solves the reference problem at the cost of introducing other problems.

Assuming that the EEG is recorded with respect to a common reference, the solution to the reference problem is to spatially analyze the data [2]. At a given sampling time, the activity of the reference modifies the signal by a spatially constant value. Spatial transformations that eliminate the effects of constant spatial shifts in the data, and hence the activity of the reference electrode, will be referred to as reference free transformations.

The most intuitive reference free transformation is to subtract the spatial average from the data. This technique, referred to as the average reference method [3], can be easily implemented on a computer when the EEG is digitally sampled. Unfortunately, it will be shown in this chapter that the average reference method introduces several non-desirable effects that have not been previously outlined in the literature.

Another reference free transformation of the data is the spatial Laplacian [4], as explained in chapter 3. The spatial Laplacian can be considered as converting the measured potentials into estimates of the radial current. Assuming that the data is sampled on a rectangular grid, a discrete approximation of the spatial Laplacian is obtained by subtracting the average value of the four surrounding electrodes from the middle electrode value. A better approach to estimating the spatial Laplacian is to utilize spherical spline Laplacians [5]. For digitally recorded data with access to computing facilities, the spatial Laplacian appears to be an attractive solution to the reference problem.

Unfortunately, the estimation of the spatial Laplacian has several problems. Firstly, the error in the estimated spatial Laplacian is affected by the location of the electrode, with the largest errors occurring at the perimeter electrodes. In addition, the spatial Laplacian will tend to emphasize artifacts generated from within the electrode array, such as electrode movement and the activity of muscles located underneath the scalp. Finally, the accurate

estimation of the spatial Laplacian requires a finer spatial sampling in order to minimize aliasing effects (discussed in chapter 5). For these reasons, the spatial Laplacian does not provide a complete, problem free solution to the reference problem.

This chapter investigates the effects of two reference free transformations on the recorded signal. The average reference transformation is analyzed for its effects on the temporal characteristics of the signal, with the results illustrating a major problem with the transformation. A new reference free transformation is proposed in order to reduce the problem associated with the average reference transformation.

4.1 Reference Free Transformations

This section considers two methods of eliminating the effects of the active reference on the measured signal based on linear transformations of the signal. One method estimates the neutral referenced signal, which by definition, is reference free. It will be shown that an accurate estimate of the neutral referenced signal requires that the actual neutral referenced signal satisfy one condition. Since the EEG does not satisfy this condition, a second method will be considered which produces a reference free residual signal.

Consider N , spatial samples of the scalp's potential function measured with respect to the potential at point P . Let $r(n)$ represent the potential at point P measured with respect to infinity (neutral referenced), where n represents the temporal sample number. The potentials measured with respect to point P can be described by

$$\vec{v}(n) = \vec{v}_-(n) - r(n)\vec{i} \quad (4.1.1)$$

where

$$\vec{v}(n) = [v_1(n), v_2(n), \dots, v_N(n)]^T,$$

$$\vec{v}_-(n) = [v_{1-}(n), v_{2-}(n), \dots, v_{N-}(n)]^T,$$

$$\vec{i} = [1, \dots, 1]^T \text{ (size } N),$$

$v_i(n)$ is the potential of channel i measured with respect to the potential at point P , $v_{i-}(n)$ is the neutral referenced potential of channel i and T is the vector transpose operator. Note that the activity of the reference shifts the measured potential function by a spatially constant, but time varying value.

Consider the problem of estimating the neutral referenced potential vector $\vec{v}_-(n)$ given the measured potential vector $\vec{v}(n)$. An estimate of the neutral referenced vector can be obtained with the aid of a linear model that consists of a constant spatial term, as given by

$$\vec{v}_-(n) \approx \vec{v}(n) - [\vec{c}^T \vec{v}(n)]\vec{i} \quad (4.1.2)$$

where $\vec{c}$ is a N , sized vector and where the scalar $\vec{c}^T \vec{v}(n)$ represents a linear combination of the measured potential vector. Using equations (4.1.1) and (4.1.2) the estimated neutral referenced signal can be written as

$$\vec{v}_-(n) = \vec{v}_-(n) - \vec{c}^T \vec{v}_-(n) \vec{i} - r(n) [1 - \vec{c}^T \vec{i}] \vec{i} \quad (4.1.3)$$

Assuming that the elements in $\vec{c}$ sum to unity, which then eliminates the effects of the reference $r(n)$, the above equation can be written as

$$\vec{v}_-(n) = \vec{v}_-(n) - [\vec{c}^T \vec{v}_-(n)] \vec{i} \quad (4.1.4)$$

It can be observed from the above equation that the estimated neutral referenced potentials are actually referenced with respect to $\vec{c}^T \vec{v}_-(n)$, which represents a linear combination of the neutral referenced potentials. The time varying reference term in (4.1.4) can be eliminated by ensuring that the vector $\vec{c}$ satisfies the relationship $\vec{c}^T \vec{v}_-(n) = 0$. In order to satisfy this relationship, one must have further information about the spatial properties of the neutral referenced signal. This approach will be further pursued in the following section.

A different approach to eliminating the effects of the active reference is to transform the measured potentials into a reference free data set. Consider the residual vector $\vec{e}(n)$, as defined by

$$\vec{e}(n) = \vec{v}(n) - \vec{v}_-(n) \quad (4.1.5)$$

Assume that $\vec{v}_-(n)$ smooths $\vec{v}(n)$ via the transformation

$$\vec{v}_-(n) = \vec{S} \vec{v}(n) \quad (4.1.6)$$

where $\vec{S}$ is the N , by N , sized similarity matrix described in chapter 3. Using equations (4.1.1), (4.1.5) and (4.1.6), the residual signal can be written as

$$\begin{aligned} \vec{e}(n) &= [\vec{I} - \vec{S}] \vec{v}(n) \\ &= [\vec{I} - \vec{S}] \vec{v}_-(n) - r(n) [\vec{i} - \vec{\beta}] \end{aligned} \quad (4.1.7)$$

where

$$\vec{\beta} = \vec{S} \vec{i}$$

$$= \begin{bmatrix} \sum_{j=1}^{N_i} S_{1j} \\ \sum_{j=1}^{N_i} S_{2j} \\ \dots \\ \sum_{j=1}^{N_i} S_{N,j} \end{bmatrix}$$

$\vec{I}$ is a N , by N , sized identity matrix and S_{ij} refers to element ij in the matrix $\vec{S}$.

Note that when $\vec{\beta} = \vec{i}$, the residual vector $\vec{e}(n)$ is independent of the reference and (4.1.7) can be written as

$$\vec{e}(n) = (\vec{I} - \vec{S})\vec{v}_-(n) \quad (4.1.8)$$

It should also be noted that a large class of linear transformations result in a similarity matrix that satisfies the condition $\vec{\beta} = \vec{i}$. In general, it can be shown that any linear model that includes a spatial constant term results in a reference free residual signal.

One should note that the characteristics of the reference free residual vector can be very different from the characteristics of either the neutral referenced or measured potential vector. Instead, the residual vector should only be considered as a linear transformation of the measured potentials. By suitably choosing the form of this linear transformation, the residual vector will hopefully extract useful information from the observed potential vector while eliminating the effects of the active reference.

4.2 Average Reference

The estimate of the neutral referenced potential vector given by (4.1.2) requires specification of the vector $\vec{c}$. It was shown that the vector $\vec{c}$ should meet two conditions; the elements in $\vec{c}$ sum to unity and that $\vec{c}^T \vec{v}_-(n) = 0$. One approach to satisfying the latter condition is to assume that the spatial average of the neutral referenced signal is zero, that is, $\vec{v}_-^T(n) \vec{i} = 0$. With this assumption, the vector $\vec{c}$ is specified by

$$\vec{c} = \frac{1}{N_s} \vec{i} \quad (4.2.1)$$

An estimate of the neutral referenced signal can then be based on equations (4.1.1) and (4.2.1), which combine to form

$$\vec{v}_-(n) = \vec{v}(n) - \left[\frac{\vec{v}^T(n) \vec{i}}{N_s} \right] \vec{i} \quad (4.2.2)$$

The above equation mathematically describes what will be referred to as the average reference transformation. This name is derived from the fact that the potential measurements are referenced to the average of the neutral referenced potentials, which can be observed by substituting (4.2.1) into (4.1.4) to form

$$\vec{v}_-(n) = \vec{v}_-(n) - \left[\frac{\vec{v}_-^T(n) \vec{i}}{N_s} \right] \vec{i} \quad (4.2.3)$$

The usefulness of this method rests upon the validity of the assumption that the spatial average of the neutral referenced signal is small.

The average reference transformation eliminates the activity of the reference electrode, while introducing the activity of the average reference. The purpose of this section is to consider if the average referenced potential measurements differ significantly from the neutral referenced potentials. In order to answer this question, the average reference transformation will be investigated using both planar and spherical models. It will be shown that the average reference transformation introduces a spatially varying bias in the temporal amplitude relationships in the data.

4.2.1 Analysis via Planar Approximation

The average reference transformation, which represents a linear transformation of the measured potential vector, can be considered as a spatial filter. In order to describe the characteristics of this spatial filter, it is convenient to assume that the EEG is sampled on a rectangular planar grid. Although the EEG is sampled on a non-rectangular grid located on a curved surface, the results that follow illustrate some of the problems that could still be expected from using the average reference transformation.

Assume the EEG is sampled by a planar rectangular 5 by 5 array where $-2 \leq x \leq 2$ and $-2 \leq y \leq 2$ define the coordinates of the spatial samples and where the measured potential function is represented by $v(x, y, n)$. Equation (4.2.3) can then be rewritten in scalar form as

$$\hat{v}_-(x, y, n) = v_-(x, y, n) - \frac{1}{25} \sum_{p=-2}^{x+2} \sum_{q=-2}^{y+2} v_-(x-p, y-q, n) \quad (4.2.4)$$

As (4.2.4) indicates, the estimated neutral referenced signal $\hat{v}_-(x, y, n)$ is independent of the reference value $r(n)$, but dependent upon the spatial average of the neutral referenced potentials. The estimated and actual neutral referenced signals will be similar when the spatial average of the neutral referenced signal approaches zero.

Removing the spatial average from the signal can be considered as a linear spatial filter. The transfer function of the spatial filter can be obtained by dividing the 2-D z-transform of (4.2.4) by the z-transform of $v_-(x, y, n)$ as follows.

$$\begin{aligned} H_{xy}(z_1, z_2) &= \frac{\hat{V}_-(z_1, z_2)}{V_-(z_1, z_2)} \\ &= 1 - \frac{1}{25} \sum_{p=-2}^{x+2} \sum_{q=-2}^{y+2} z_1^{-p} z_2^{-q} \end{aligned} \quad (4.2.5)$$

The important feature of (4.2.5) is that $H_{xy}(z_1, z_2)$ is shift variant due to the inclusion of the xy terms in the summations. This shift variant transfer function, which results in a different transfer function at each sampling location, is the source of the problem with the average reference transformation.

Using (4.2.5), the spatial wavenumber response at each spatial sample for the average reference transformation is given by

$$H_{xy}(k_x, k_y) = 1 - \frac{1}{25} \sum_{p=x-2}^{x+2} \sum_{q=y-2}^{y+2} \exp[-jk_x p - jk_y q] \quad (4.2.6)$$

where k_x and k_y represent the normalized wavenumbers. Contour plots of $|H_{xy}(k_x, k_y)|^2$ are illustrated in Figure 4.1 for a corner sample $\{x = -2, y = -2\}$ and for the center sample $\{x = 0, y = 0\}$. Each response resembles the characteristics of a spatial high-pass filter, with zero gain at the origin and approximately unity gain at higher wavenumbers. However, note that the two responses are different, with the corner samples response being sharper and more directional than the center samples response. It is this difference in the wavenumber responses that causes problems with the average reference transformation.

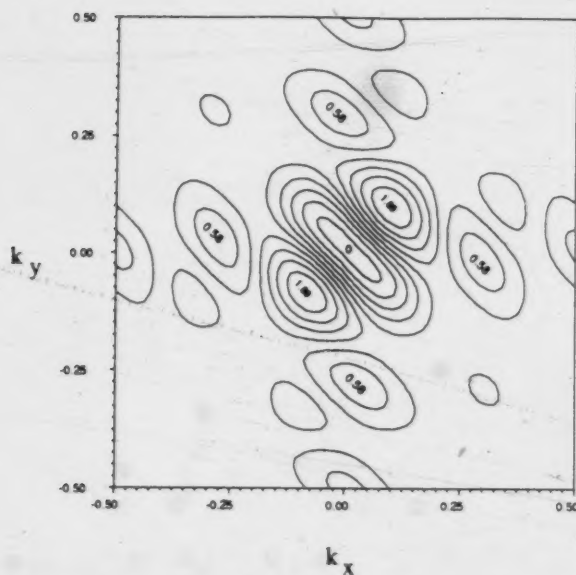
The next example considers a spatially band-limited neutral referenced signal sampled by the same 5 by 5 array. Assume that the signal's spatial power spectrum is constant in the region $k_x^2 + k_y^2 \leq R^2$ and zero otherwise, resulting in a circularly spatially bandlimited signal. Also, assume that the temporal variance of each channel is unity and that the signal is sampled over a long time interval. The temporal variance of each average referenced channel ($\sigma_{xy}^2(R)$) can be determined, with the aid of Parseval's theorem, via

$$\sigma_{xy}^2(R) = \frac{1}{25} \int_{k_x=-R}^R \int_{k_y=-\sqrt{R^2-k_x^2}}^{\sqrt{R^2-k_x^2}} |H_{xy}(k_x, k_y)|^2 dk_x dk_y \quad (4.2.7)$$

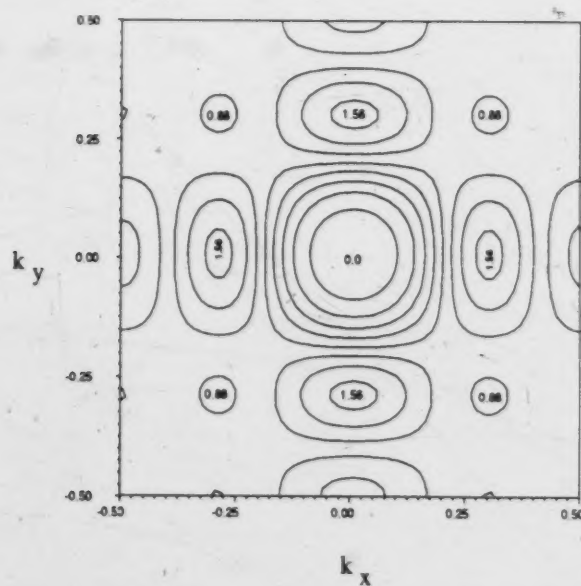
Table 4.1 tabulates $\sigma_{xy}^2(R)$ for $R = \pi/4$ and $R = \pi/2$. Ideally, the temporal variances of the spatial samples should be equal after the spatial average is removed. However, note from the results given in Table 4.1 that the corner samples' variances are larger than the center sample's variance due to the formers sharper wavenumber response. Increasing the signals spatial bandwidth from $R = \pi/4$ to $R = \pi/2$ results in a more even distribution of the variances.

It is not surprising that the average reference transformation has problems with signals composed of low wavenumbers when one considers the wavelength and the dimensions of the sampling grid. As the wavenumber decreases to zero, the wavelength becomes larger than the dimensions of the sampling grid which results in only a portion of the wave being sampled. Although the spatial average of the wave is zero, portions of the wave do not have zero average. Since the average reference transformation assumes that the spatial average of the reference free signal is zero, it will provide an incorrect estimate of the reference for signals composed of low wavenumbers.

The main point to be noted from the planar analysis of the average reference transformation is that signals composed of low wavenumbers sampled at a limited number of locations result in a reduction of the temporal variance for centrally located samples when compared to the variance of perimeter sample locations. It can also be shown that the phase relationships between the average referenced electrodes is different than the phase relationships between the neutral referenced electrodes.



(a) corner sample



(b) center sample

Figure 4.1 - Contour plot of $|H_{ij}(k_x, k_y)|^2$ vs. spatial wavenumber for corner and central samples. These plots illustrate the spatial wavenumber response of the average reference transformation for a signal sampled by a 5 by 5 rectangular array of electrodes. Note that the corner samples response is sharper and more directional than the center samples response.

0.763	0.549	0.464	0.549	0.763
0.549	0.297	0.200	0.297	0.549
0.464	0.200	0.102	0.200	0.464
0.549	0.297	0.200	0.297	0.549
0.763	0.549	0.464	0.549	0.76

(a) $R = \pi/4$

0.971	0.879	0.848	0.879	0.971
0.879	0.735	0.685	0.735	0.879
0.848	0.685	0.626	0.685	0.848
0.879	0.735	0.685	0.735	0.879
0.971	0.879	0.848	0.879	0.971

(b) $R = \pi/2$

Table 4.1 - Tabulations of $\sigma_{\eta}^2(R)$ for an average referenced bandlimited signal. The signal is assumed to be circularly bandlimited to the region $k_x^2 + k_y^2 \leq R^2$. The mean and standard deviation of the values indicated in table (a) is 0.451 ± 0.197 and 0.799 ± 0.107 for the values in table (b). These tables illustrate the effect on the temporal variance of a channel due to removing the spatial average. Note that the variance is smallest for the centrally located channel and largest for the perimeter channels. This effect is reduced when the spatial bandwidth of the signal R is increased from $\pi/4$ to $\pi/2$.

4.2.2 Analysis via Spherical Approximation

The average reference transformation, as mentioned in the previous section, assumes that the spatial average of the neutral referenced signal is zero. Thus, an estimate of the neutral referenced signal can be obtained by removing the spatial average from the data. Justification of this assumption is based upon a spherical model of the human head.

Consider a set of current generators located in a homogeneous conducting sphere. Assume that the potential function on the outer surface of the sphere is known and is referenced to the potential at point P . It has been shown [6] that the average value of the potential function over the surface of a sphere, referenced to point P , is equal to the potential of point P with respect to infinity (neutral reference). Thus, the potential function measured with respect to point P can be referenced to a neutral point by removing the spatial average from the function. In other words, the spatial average of the neutral referenced signal is zero. The justification of the average reference transformation is based on this result.

The human head can be approximated by a conducting sphere with internal current generators, as described in chapter 2. With this assumption, the EEG is measured on the outer surface of the sphere with respect to the potential at point P . By sampling the potential function at a sufficient number of points over the surface of the sphere, the average value of the sampled potentials would provide a good estimate of the neutral referenced potential at point P . In this situation, the average reference transformation would then provide a good estimate of the neutral referenced potentials.

Unfortunately, the actual EEG is sampled over approximately the surface of a hemisphere. The assumption that the spatial average of the neutral referenced signal is zero is no longer valid which brings into question the validity of the average reference transformation. In addition, the non-spherical geometry and the inhomogeneity of the head along with the effects of the neck cause further problems with the transformation.

In order to investigate the effect the average reference transformation on spherically sampled data, assume that the potential function is known only over the region defined by a spherical cap. The average reference estimate of $r(n)$ is given by

$$\hat{r}(n) = \frac{1}{2\pi(1 - \cos(\phi_0))} \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} v \sin \phi d\phi d\theta \quad (4.2.8)$$

where v is the spherical potential function and ϕ_0 defines the size of the spherical cap. The above reference estimate is unbiased only when the potential function is known over the whole surface of the sphere ($\phi_0 = \pi$). Representing the potential function v by its spherical harmonic expansion, (4.2.8) can be expressed as

$$\hat{r}(n) = \frac{1}{2\pi(1 - \cos \phi_0)} \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} \sum_{q=0}^{\infty} \sum_{p=-q}^q a_q^p Y_q^p \sin \phi d\phi d\theta \quad (4.2.9)$$

Using equation (3.3.1), the results given in (3.3.5) and the fact that $Y_0^0 = \frac{1}{2\sqrt{\pi}}$, the above equation can be simplified to

$$\hat{r}(n) = \frac{1}{\sqrt{\pi}(1 - \cos \phi_0)} \sum_{q=0}^{\infty} a_q^0 R(0, 0; 0, q; \phi_0) \quad (4.2.10)$$

Now consider the situation when the function v is only known on the surface of a hemisphere ($\phi_0 = \pi/2$). Using the results shown in Table 3.1, equation (4.2.10) can be written as

$$\hat{r}(n) = \frac{1}{2\sqrt{\pi}} (a_0^0 + 0.866a_1^0 - 0.332a_3^0 + 0.208a_5^0 \pm \dots) \quad (4.2.11)$$

The average value of the potential function v over the whole sphere due to the level of the reference (which is the true value for $r(n)$) is directly related to a_0^0 and is given by

$$\begin{aligned} r(n) &= \frac{1}{2\pi(1 - \cos \phi_0)} \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} a_0^0 Y_0^0 \sin \phi d\phi d\theta \\ &= \frac{a_0^0}{2\sqrt{\pi}} \end{aligned} \quad (4.2.12)$$

By using (4.2.11) and (4.2.12), the average reference estimate for $r(n)$ is thus given by

$$\hat{r}(n) = r(n) + \frac{1}{2\sqrt{\pi}} (0.866a_1^0 - 0.332a_3^0 + 0.208a_5^0 \pm \dots) \quad (4.2.13)$$

The bias in the average reference estimate for $r(n)$ is determined by the relative magnitude of the zero order spherical harmonic coefficients of the potential function.

It can be observed from (4.2.13) that the largest difference between the estimated and actual reference levels will occur when the signal is dominated by zero order, low degree spherical harmonic functions. The magnitude of the zero order spherical harmonic waves is largest near the pole $\phi_0 = 0$, which corresponds to the center point of the hemispherical surface. Removing the spatial average from the function will then decrease the magnitude of the function most significantly in the center. For hemispherically sampled data, one would again expect that the temporal variance of centrally located samples would decrease in relationship to the temporal variance of perimeter samples.

4.2.3 Additional Problems with the Average Reference Transformation

It is worth noting several additional problems related to the average reference transformation. These problems have been previously noted by Lehmann [7] and are repeated here for completeness.

Assume that one or more of the channels are 'bad', due to such problems as poor electrode contact and localized muscle activity. By using all of the electrodes in the calculation of the reference, this bad electrode will affect the estimate of the reference. Since the average reference is removed from all the channels, a bad channel's signal can propagate

into all of the channels. A localized problem has been converted to a global problem. This problem could be overcome by either ensuring artifact-free data or by rejecting bad channels from the calculation of the spatial average.

Another problem of the average reference transformation is related to the fact that the actual data is non-uniformly sampled. In order to calculate the spatial average, a weighting function should be defined that describes the percentage of the surface that is defined by the sample location. Isolated electrodes would be weighted higher than closely spaced electrodes. The spatial average is then calculated as the weighted sum of the electrode values.

A final problem is that frontal activity increases when alpha dominant data is average referenced. Normally, alpha activity is strongest in the occipital region and weakest in the frontal regions. It has been found that the average reference transformation increases the amount of alpha activity in the frontal regions, when compared to occipital regions. Alpha activity is dominated by low wavenumbers which increases the error in the estimated reference value, as described in this chapter. This problem has also been referred to as the dipole effect.

4.3 Perimeter Reference

The previous section showed that the average reference tends to remove more variance from signals recorded in centrally located regions as compared to perimeter regions. In order to offset this effect, an intuitive approach would be to base the reference estimate only on the activity of the perimeter samples. This approach has not been published in the EEG literature. This section analyzes the perimeter reference method for both planar and spherical sampling using the same approach as in the last section.

In order to illustrate the effect of using a perimeter reference, again assume that a spatially bandlimited signal is sampled by a planar 5 by 5 array. The estimate of the reference free signal based on the perimeter reference is given by

$$\hat{v}_-(x, y, n) = v_-(x, y, n) - \frac{1}{16} \sum_p \sum_q v_-(p, q, n) \quad (4.3.1)$$

where the summations only involves the 16 perimeter channels. The channels' variance for the perimeter referenced signal as defined by (4.2.7) is given in Table 4.2 for $R = \pi/4$ and $R = \pi/2$. By comparing the average reference results in Table 4.1 with the perimeter reference results in Table 4.2, it can be noted that the standard deviation of the latter results is smaller. This infers that the perimeter reference provides a better estimate of the reference when compared to the average reference.

Next, consider a continuous potential function observed on the surface of a hemisphere. The perimeter reference estimate is defined as the average value of the spherical function v on a great circle path with $\phi = \pi/2$, which can be calculated by

$$\hat{r}(n) = \frac{1}{2\pi} \int_{\theta=0}^{2\pi} v d\theta \quad (4.3.2)$$

By using the spherical harmonic expansion of the potential function, the above integral can be simplified to

$$\begin{aligned}\hat{r}(n) &= \frac{1}{2\sqrt{\pi}} \sum_{q=0}^{\infty} a_{2q}^0 \sqrt{4q+1} P_{2q}(\pi/2) \\ &= r(n) + \frac{1}{2\sqrt{\pi}} \left(-\frac{1}{2} a_2^0 + \frac{3}{8} a_4^0 - \frac{5}{16} a_6^0 \pm \dots \right)\end{aligned}\quad (4.3.3)$$

One can note that the reference estimate is corrupted by the signal's spherical harmonic coefficients of order zero and even degree while the average reference estimate given in (4.2.13) is affected by the zero order, odd degree coefficients.

In order to compare the quality of the average and perimeter reference estimates given by (4.2.13) and (4.3.3), one must know the spherical harmonic expansion of the potential function v . The following chapter indicates that the spatial spectrum of the EEG is a monotonically decreasing function for decreasing wavelengths. Using the relationship given in (3.2.11) that defines the spectrum in terms of the spherical harmonic coefficients, it is reasonable to assume that $|a_n^0|$ is also a monotonically decreasing function for increasing values of n . Assuming that the first term in the infinite series in (4.2.12) and (4.3.3) is much larger than the remaining terms, the perimeter reference should provide a better estimate of the reference.

For the 31-electrode montage shown in Figure 2.4, an estimate of the neutral referenced EEG using the perimeter reference is given by

$$\hat{v}_{i-}(n) = v_i(n) - \frac{1}{14} \sum_k v_k(n) \quad (4.3.4)$$

$$k = \{2, 12, 14, 24, 16, 8, 10, 9, 7, 15, 23, 13, 11, 1\}$$

It was found that the perimeter electrodes were approximately equispaced if channels 20 and 30 were excluded, thus the perimeter reference is the average of the perimeter channel measurements.

0.707	0.537	0.473	0.537	0.707
0.537	0.338	0.264	0.338	0.537
0.473	0.264	0.187	0.264	0.473
0.537	0.338	0.264	0.338	0.537
0.707	0.537	0.473	0.537	0.707

(a) $R = \pi/4$

0.850	0.834	0.856	0.834	0.850
0.834	0.889	0.952	0.889	0.834
0.856	0.952	1.030	0.952	0.856
0.834	0.889	0.952	0.889	0.834
0.850	0.834	0.856	0.834	0.850

(b) $R = \pi/2$

Table 4.2 - Tabulations of $\sigma_{\eta}^2(R)$ for a perimeter referenced bandlimited signal. The signal is assumed to be circularly bandlimited to the region $k_x^2 + k_y^2 \leq R^2$. The mean and standard deviation of the values indicated in table (a) is 0.457 ± 0.156 and 0.834 ± 0.067 for the values in table (b). These results illustrate the effect on the temporal variance of removing the perimeter average from a signal sampled by a 5 by 5 array. Note that the variance is smaller for central vs. perimeter channels when the spatial bandwidth is $\pi/4$. This variation is reduced and inverted when the spatial bandwidth of the signal is increased to $\pi/2$.

4.4 Experimental Results

This section investigates the effect of using the average and perimeter reference on recorded EEGs. Theoretical results indicate that the average reference will remove more variance from centrally located channels, as compared to perimeter channels. The theoretical results also showed that the perimeter reference should decrease this effect.

The experimental data consisted of the 15 subjects' EEGs recorded with the reference electrode placed on the left-ear lobe, under the eyes closed (EC) and eyes positioned (EP) conditions as explained in chapter 2. Recall that three 8.54-second epochs were selected from each EEG record, resulting in a two 45 epoch sets. Each of the epochs were then phase and magnitude corrected.

The spatial distribution of the variance for the left-ear, average and perimeter referenced data sets were determined as follows

$$\sigma_i^2 = \frac{\sum_{n=1}^{960} u_i^2(n)}{\sum_{i=1}^{31} \sum_{n=1}^{960} u_i^2(n)} \quad (4.4.1)$$

where $u_i(n)$ is either the left-ear, average or perimeter referenced data for channel i . The parameter σ_i^2 calculates the portion of the epoch's variance accounted for by channel i and satisfies the relationship $\sum_{i=1}^{31} \sigma_i^2 = 1$. The mean and standard deviation of σ_i^2 are shown in Tables 4.3-4.4, Tables 4.5-4.6 and Tables 4.7-4.8 for the left-ear, average and perimeter referenced data sets respectively. Associated with each table is a bar plot of the mean values of σ_i^2 .

The second parameter calculated for each epoch was the normalized channel variance NCV_i and is defined by

$$NCV_i = \frac{\sum_{n=1}^{960} \hat{v}_{i,-}^2(n)}{\sum_{n=1}^{960} v_i^2(n)} \quad (4.4.2)$$

where $\hat{v}_{i,-}(n)$ is the estimated neutral referenced potential of channel i obtained by either the average or perimeter reference transformations and $v_i(n)$ is the left-ear referenced potential. This parameter describes the ratio of the channel variance of the estimated neutral referenced signal over the channel variance of the original, left-ear referenced signal. The mean and standard deviation of NCV_i are shown in Tables 4.9-4.10 and Tables 4.11-4.12 for the average and perimeter referenced data sets respectively.

Nasion

	2.2 $\pm$ 0.6	2.3 $\pm$ 0.6	2.3 $\pm$ 0.6	
1.7 $\pm$ 0.5	2.5 $\pm$ 0.7	2.7 $\pm$ 0.8	2.9 $\pm$ 0.9	2.4 $\pm$ 0.7
1.1 $\pm$ 0.4	2.7 $\pm$ 0.8	3.4 $\pm$ 1.0	3.0 $\pm$ 0.9	2.1 $\pm$ 0.7
0.9 $\pm$ 0.4	2.5 $\pm$ 0.5	3.5 $\pm$ 0.8	3.3 $\pm$ 0.8	1.9 $\pm$ 0.7
1.5 $\pm$ 0.5	3.1 $\pm$ 0.4	4.0 $\pm$ 0.7	3.9 $\pm$ 0.6	2.6 $\pm$ 1.0
4.1 $\pm$ 2.2	4.7 $\pm$ 1.7	6.2 $\pm$ 2.7	5.7 $\pm$ 1.5	5.8 $\pm$ 2.1
	4.7 $\pm$ 1.6	4.6 $\pm$ 1.6	5.9 $\pm$ 2.7	

Inion

Table 4.3 - Mean and standard deviation of $\sigma_i^2 \times 100$ for the left-ear referenced EC data set. The parameter σ_i^2 illustrates the spatial distribution of the signals' temporal variance. The spatial organization of the table shows the approximate locations of the channels (refer to Figure 2.4). Note the reduction in the variance for those channels located near the left ear.

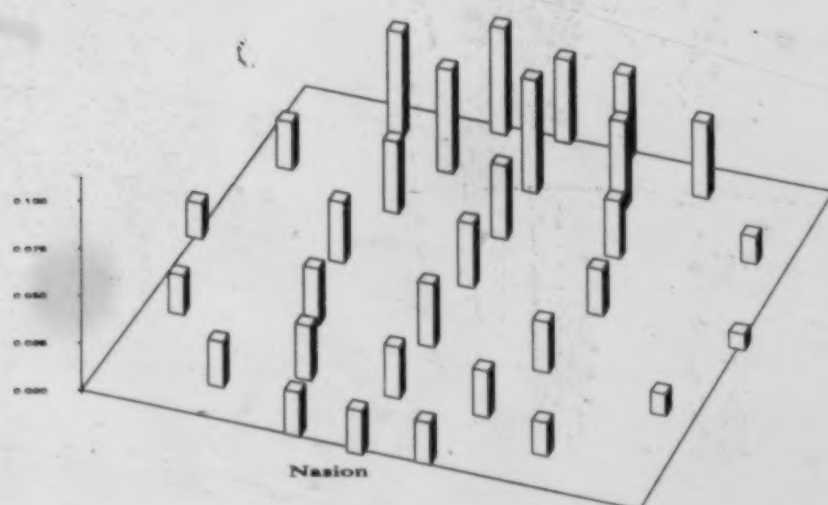


Figure 4.2 - Spatial plot of the mean values of σ_i^2 contained in Table 4.3 along with a more realistic spatial projection of the channel locations.

	2.9 ± 0.6	3.0 ± 0.5	3.0 ± 0.6	
2.3 ± 0.6	3.2 ± 0.5	3.5 ± 0.6	3.8 ± 0.6	3.2 ± 0.6
1.5 ± 0.4	3.5 ± 0.6	4.6 ± 0.8	4.0 ± 0.5	2.8 ± 0.5
1.3 ± 0.4	3.3 ± 0.8	4.3 ± 0.6	4.1 ± 0.6	2.7 ± 0.8
1.6 ± 0.3	3.2 ± 0.5	4.1 ± 0.6	3.9 ± 0.5	2.8 ± 0.4
2.5 ± 0.7	3.3 ± 0.8	4.2 ± 1.3	4.0 ± 0.6	3.4 ± 0.6
	3.0 ± 0.9	3.2 ± 1.1	3.6 ± 1.4	

Table 4.4 - Mean and standard deviation of $\sigma^2 \times 100$ for the left-ear referenced EP data set.

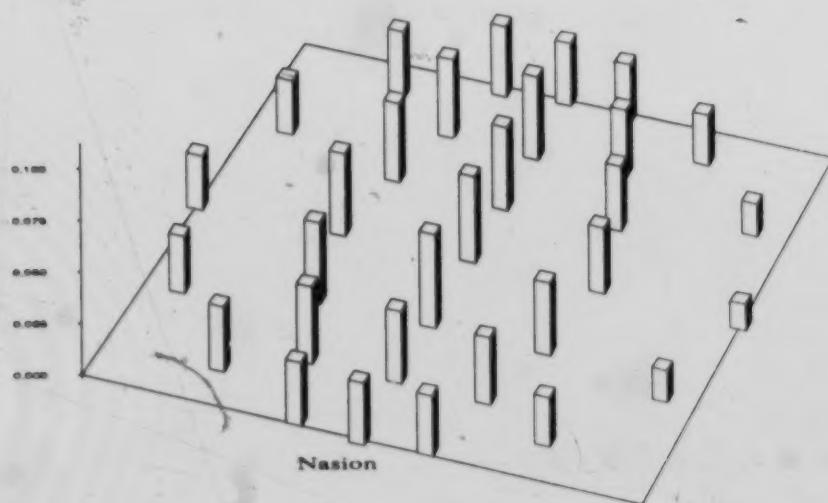


Figure 4.3 - Spatial plot of the mean values of σ^2 contained in Table 4.4.

	2.8 ± 0.7	2.7 ± 0.6	2.8 ± 0.6	
2.7 ± 0.7	2.4 ± 0.5	2.5 ± 0.5	2.6 ± 0.5	2.9 ± 0.7
2.2 ± 0.7	2.1 ± 0.4	2.4 ± 0.8	2.0 ± 0.6	2.4 ± 0.6
2.1 ± 0.9	1.6 ± 0.5	1.8 ± 0.6	1.7 ± 0.6	2.2 ± 0.8
2.8 ± 0.9	1.9 ± 0.5	2.0 ± 0.7	2.1 ± 0.6	2.9 ± 1.8
6.1 ± 2.2	4.5 ± 1.6	5.4 ± 3.1	5.0 ± 1.7	7.4 ± 2.9
	5.7 ± 2.1	5.0 ± 1.6	6.9 ± 3.1	

Table 4.5 - Mean and standard deviation of $\sigma_i^2 \times 100$ for the average referenced EC data set.

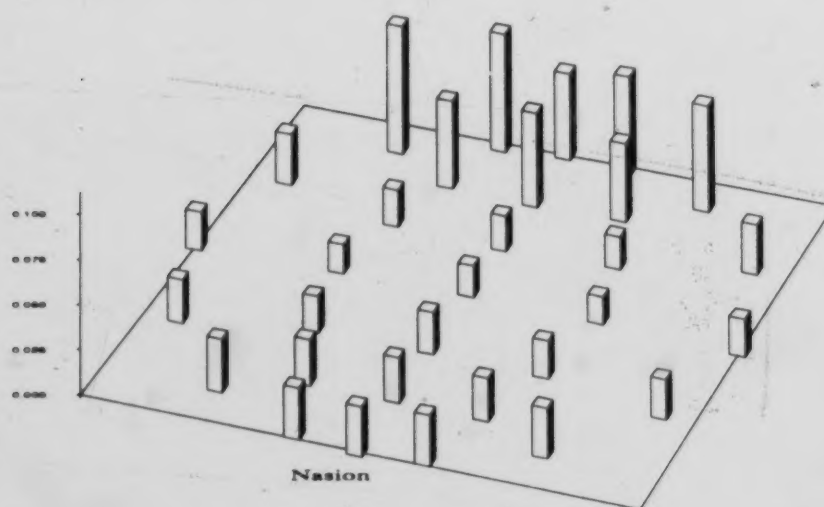


Figure 4.4 - Spatial plot of the mean values of σ_i^2 contained in Table 4.5

	3.5 ± 0.7	3.3 ± 0.7	3.6 ± 0.8	
3.7 ± 0.8	2.8 ± 0.4	2.9 ± 0.5	3.0 ± 0.4	3.9 ± 0.9
3.0 ± 0.6	2.6 ± 0.6	3.3 ± 0.8	2.5 ± 0.5	3.1 ± 0.5
3.2 ± 0.9	2.3 ± 1.3	2.6 ± 0.7	2.4 ± 0.9	3.2 ± 1.7
3.5 ± 0.8	2.2 ± 0.5	2.4 ± 0.6	2.4 ± 0.7	3.0 ± 0.7
4.3 ± 0.9	3.3 ± 0.9	3.8 ± 1.5	3.5 ± 0.7	4.3 ± 1.1
	4.0 ± 1.0	3.9 ± 1.1	4.4 ± 1.6	

Table 4.6 - Mean and standard deviation of $\sigma_i^2 \times 100$ for the average referenced EP data set.

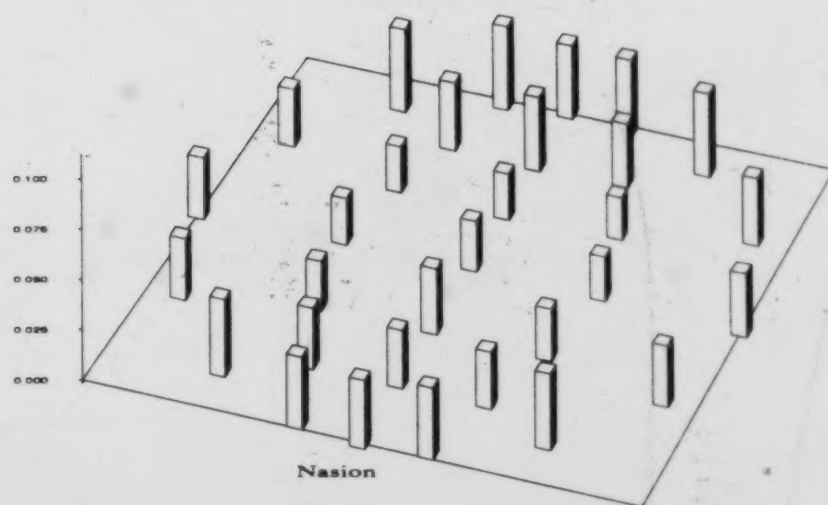


Figure 4.5 - Spatial plot of the mean values of σ_i^2 contained in Table 4.6.

	2.7 ± 0.5	2.7 ± 0.5	2.7 ± 0.4	
2.4 ± 0.6	2.7 ± 0.5	2.9 ± 0.5	2.9 ± 0.6	2.6 ± 0.6
1.8 ± 0.5	2.8 ± 0.7	3.4 ± 1.2	2.7 ± 0.8	2.0 ± 0.5
1.5 ± 0.6	2.3 ± 0.7	3.1 ± 0.9	2.4 ± 0.8	1.6 ± 0.5
2.0 ± 0.6	2.5 ± 0.6	3.3 ± 0.9	2.7 ± 0.7	2.1 ± 1.3
5.0 ± 2.1	4.6 ± 1.5	6.0 ± 3.2	5.1 ± 1.7	6.1 ± 2.5
	5.0 ± 1.7	4.5 ± 1.4	5.9 ± 2.7	

Table 4.7 - Mean and standard deviation of $\sigma_i^2 \times 100$ for the perimeter referenced EC data set.

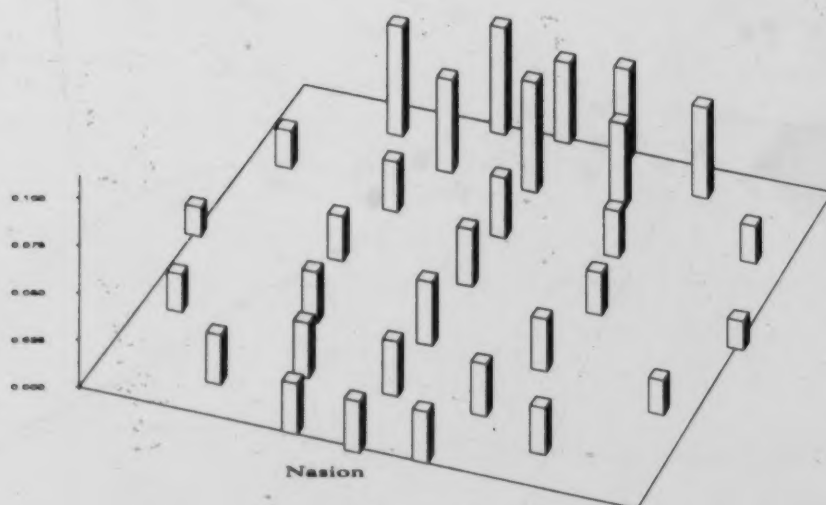


Figure 4.6 - Spatial plot of the mean values of σ_i^2 contained in Table 4.7.

	3.0 ± 0.6	3.0 ± 0.6	3.0 ± 0.6	
2.9 ± 0.7	3.2 ± 0.5	3.3 ± 0.7	3.4 ± 0.6	3.1 ± 0.7
2.2 ± 0.4	3.5 ± 0.7	4.8 ± 1.1	3.4 ± 0.7	2.3 ± 0.4
2.2 ± 0.6	3.2 ± 1.4	4.1 ± 0.9	3.3 ± 0.9	2.2 ± 1.3
2.5 ± 0.5	3.0 ± 0.6	3.8 ± 0.8	3.2 ± 0.7	2.2 ± 0.5
3.4 ± 0.8	3.5 ± 0.9	4.3 ± 1.6	3.7 ± 0.7	3.5 ± 0.9
	3.5 ± 0.9	3.4 ± 1.2	3.8 ± 1.5	

Table 4.8 - Mean and standard deviation of $\sigma_i^2 \times 100$ for the perimeter referenced EP data set.

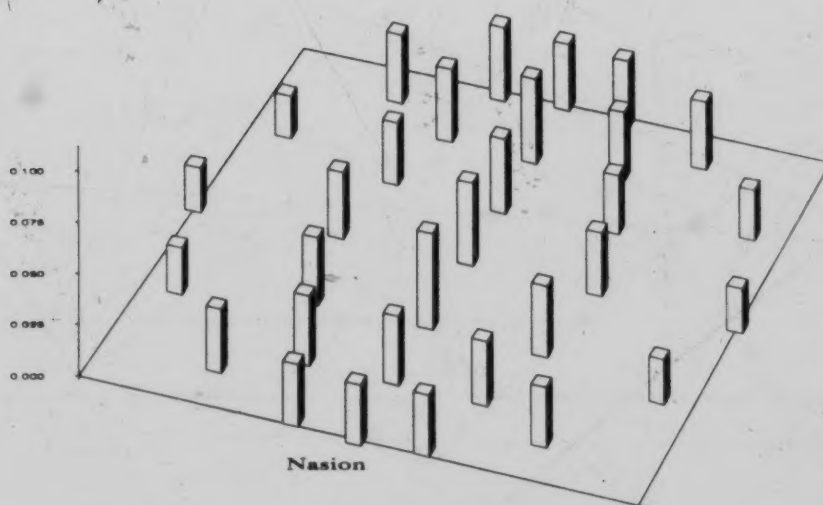


Figure 4.7 - Spatial plot of the mean values of σ_i^2 contained in Table 4.8.

	0.70 ± 0.29	0.66 ± 0.22	0.67 ± 0.27	
0.87 ± 0.36	0.55 ± 0.24	0.52 ± 0.19	0.50 ± 0.21	0.66 ± 0.24
1.10 ± 0.47	0.43 ± 0.16	0.37 ± 0.13	0.37 ± 0.15	0.63 ± 0.23
1.35 ± 0.55	0.32 ± 0.11	0.26 ± 0.08	0.26 ± 0.09	0.59 ± 0.18
1.01 ± 0.34	0.30 ± 0.08	0.25 ± 0.05	0.27 ± 0.06	0.53 ± 0.12
0.81 ± 0.18	0.49 ± 0.12	0.44 ± 0.10	0.44 ± 0.09	0.64 ± 0.12
	0.63 ± 0.15	0.56 ± 0.12	0.60 ± 0.15	

Table 4.9 - Mean and standard deviation of NCV_i for the average referenced EC data set. The parameter NCV_i indicates the ratio of the variance of the average referenced channel i over the variance of the left-ear referenced channel. Note that NCV_i is smallest in the central regions and largest near the left ear.

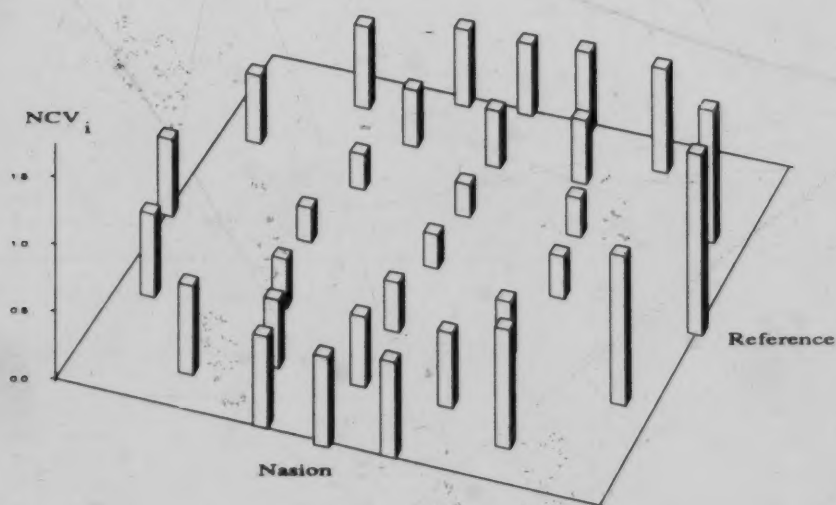


Figure 4.8 - Spatial plot of the mean values of NCV_i shown in Table 4.9.

	0.48 ± 0.16	0.45 ± 0.15	0.47 ± 0.16	
0.64 ± 0.19	0.35 ± 0.12	0.33 ± 0.12	0.32 ± 0.11	0.47 ± 0.13
0.81 ± 0.26	0.29 ± 0.09	0.28 ± 0.07	0.25 ± 0.09	0.44 ± 0.14
1.05 ± 0.41	0.26 ± 0.11	0.23 ± 0.06	0.23 ± 0.09	0.45 ± 0.13
0.87 ± 0.31	0.27 ± 0.08	0.22 ± 0.06	0.24 ± 0.08	0.42 ± 0.12
0.68 ± 0.18	0.39 ± 0.11	0.35 ± 0.09	0.34 ± 0.07	0.48 ± 0.11
	0.52 ± 0.14	0.47 ± 0.11	0.48 ± 0.12	

Table 4.10 - Mean and standard deviation of NCV_i for the average referenced EP data set.

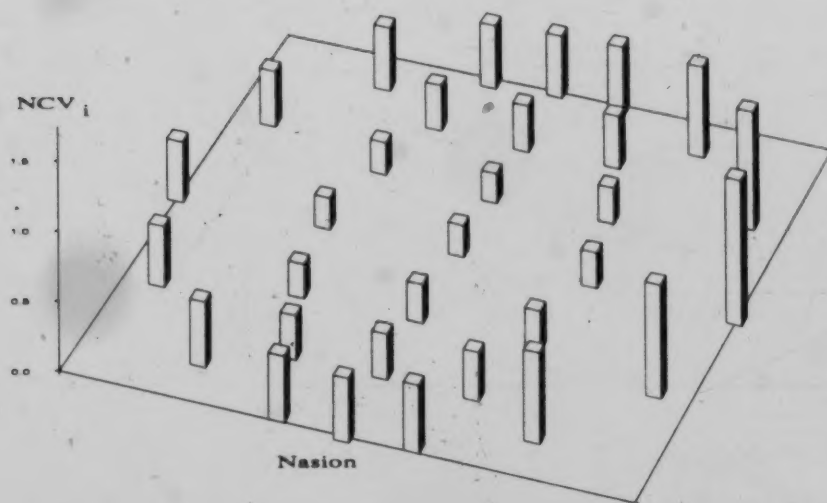


Figure 4.9 - Spatial plot of the mean values of NCV_i contained in Table 4.10.

	0.74 ± 0.29	0.71 ± 0.28	0.70 ± 0.28	
0.86 ± 0.36	0.67 ± 0.25	0.65 ± 0.23	0.61 ± 0.22	0.66 ± 0.25
1.01 ± 0.46	0.61 ± 0.19	0.57 ± 0.17	0.52 ± 0.18	0.59 ± 0.25
1.08 ± 0.42	0.50 ± 0.14	0.49 ± 0.13	0.41 ± 0.12	0.48 ± 0.18
0.78 ± 0.24	0.45 ± 0.09	0.45 ± 0.09	0.39 ± 0.07	0.41 ± 0.09
0.71 ± 0.13	0.55 ± 0.10	0.53 ± 0.11	0.49 ± 0.09	0.57 ± 0.09
	0.60 ± 0.12	0.55 ± 0.12	0.56 ± 0.12	

Table 4.11 - Mean and standard deviation of NCV_i for the perimeter referenced EC data set. For this table, NCV_i is the ratio of the variance of the perimeter referenced channel over the variance of the left-ear referenced channel.

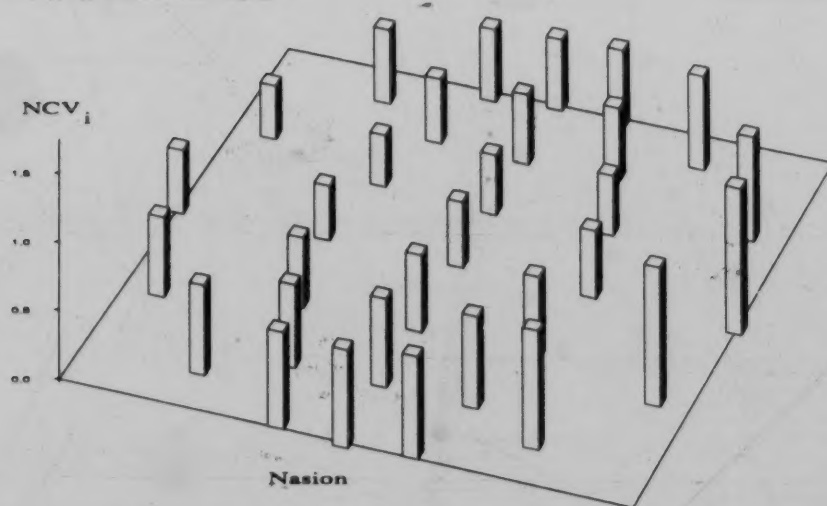


Figure 4.10 -Spatial plot of the mean values of NCV_i shown in Table 4.11.

	0.46 ± 0.14	0.44 ± 0.14	0.44 ± 0.14	
0.56 ± 0.16	0.43 ± 0.13	0.42 ± 0.12	0.40 ± 0.12	0.43 ± 0.11
0.65 ± 0.20	0.43 ± 0.11	0.45 ± 0.09	0.37 ± 0.11	0.38 ± 0.12
0.80 ± 0.30	0.42 ± 0.13	0.42 ± 0.10	0.36 ± 0.11	0.36 ± 0.10
0.70 ± 0.24	0.40 ± 0.11	0.40 ± 0.10	0.35 ± 0.10	0.34 ± 0.09
0.61 ± 0.15	0.46 ± 0.11	0.45 ± 0.10	0.41 ± 0.08	0.44 ± 0.09
	0.51 ± 0.12	0.47 ± 0.10	0.47 ± 0.10	

Table 4.12 - Mean and standard deviation of NCV_i for the perimeter referenced EP data set.

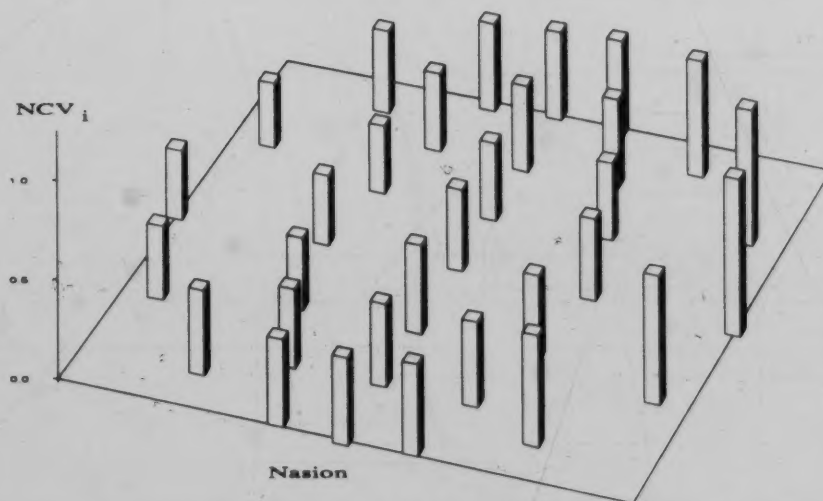


Figure 4.11 - Spatial plot of the mean values of NCV_i contained in Table 4.12.

4.5 Discussion of Results

From the left-ear referenced data results shown in Tables 4.3-4.4, it can be noted that those channels located near the reference point (left-ear) have a reduced temporal variance, as compared to the remaining channels. This effect can be explained by assuming that the signal is spatially bandlimited, which will result in a significant correlation between the potentials of adjacent channels. Potentials measured near the reference point will be reduced due to the correlation with the reference activity. This effect is significantly reduced for the average and perimeter referenced data set results shown in Tables 4.5-4.8.

It can also be noted from the EC data set results shown in Tables 4.3, 4.5 and 4.7 that the channel variance is highest in the occipital region. It is well-known that EC recordings are often associated with the presence of alpha activity, which is strongest in the occipital region. Inspection of the EC EEG records confirmed the presence of alpha activity.

The NCV_i results in Tables 4.9-4.10 for the average referenced data show that the reduction in the channels' variance is more pronounced for centrally located channels. A more even reduction in the channels' variance can be noted from the perimeter reference data sets results shown in Tables 4.11-4.12. Another effect that can be noted from the NCV_i results shown in Tables 4.9-4.12 is the relative increase in the variance of channels located near the left ear. This result is again due to the effect of the left-ear reference point, which reduces the variance of the original potentials measured near the left ear. Since the parameter NCV_i is normalized by the original channels' variance, this will result in an increased value for NCV_i for channels located near the left-ear. As the channels' distance from the left-ear is increased, this effect becomes less pronounced.

In comparing the theoretical $\sigma_{n_i}^2(R)$ and experimental NCV_i results, it should be noted that the parameters have different definitions. The parameter NCV_i is normalized by the channel variance of the left ear referenced data while $\sigma_{n_i}^2(R)$ is normalized by the variance of the neutral referenced data. As was previously noted, NCV_i will be artificially increased for those channels located near the left-ear reference point. In addition, one would expect the variance of the left-ear referenced signal to be higher than the neutral referenced signal since the left-ear referenced signal includes a spatial constant term due to the activity of the reference. The increased variance in the measured signal will result in the experimental NCV_i values being smaller than the theoretical $\sigma_{n_i}^2(R)$.

Taking the above points into consideration, there is a reasonable correspondence between the theoretical and experimental results for the average and perimeter referenced data. Average referencing the data tends to remove more of the signal's variance from centrally located channels, which suggests that the average referenced signal is a biased estimate of the neutral referenced signal. Since this effect was reduced in both the theoretical and experimental results for the perimeter referenced data, one can conclude that the perimeter referenced signal provides a better estimate of the neutral referenced signal.

The advantage of using either the average or perimeter reference is that it reduces the effects of the active reference. The reduction in the temporal variance for those channels located near the reference channel is eliminated. However, both the theoretical and experimental results indicate that the average reference introduces a non-desirable effect into the data. Although the perimeter reference reduces this problem, the perimeter reference is not equivalent to a neutral reference. Thus the advantages of using either the average or perimeter reference must be weighed against the disadvantages.

A better estimate of the neutral referenced signal might be obtained with a more involved model for the spatial EEG. For example, the spatial EEG could be approximated by either a truncated spherical harmonic expansion or a spherical spline (to be described in chapter 6). Both of these models includes a spatially constant term which could be used to estimate the reference level. The spatially constant component of the model could then be removed, resulting in an estimate of the neutral referenced signal.

The major problem with the above approach is that the accuracy of the estimated spatial constant becomes more sensitive to errors in the spherical approximation as the model becomes more involved. Since the average and perimeter reference transformations make minimal assumptions about the data, these methods are not particularly sensitive to deviations from spherical geometry. However, one would expect a greater sensitivity in the model parameters to deviations from the assumed geometry with a more complicated model such as a spherical spline. Thus, a more involved model might actually produce a poorer estimate of the neutral referenced signal, as compared to either the average or the perimeter reference transformation.

An alternate approach to eliminating the effects of the active reference is to calculate the residual signal, as explained previously in this chapter. The residual signal was defined as the difference between the original and a spatially smoothed version of the data. Although the residual signal is ensured to be reference free, it does not attempt to estimate the neutral referenced potentials. Instead, the residual signal can be considered as a spatial high-pass filtered version of the original data.

4.6 Conclusions

The results in this chapter showed that the average reference transformation biases the temporal characteristics of the EEG. In order to reduce this effect, the perimeter reference transformation was proposed. Both theoretical and experimental results indicated that the perimeter reference transformation provides a better estimate of the neutral referenced signal than the average reference transformation.

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5 Estimating the Spatial Bandwidth of the EEG

One of the most basic problems in the spatial analysis of the EEG is the limited number of channels used to sample the signal. Most clinical EEG systems are limited to 20 channel recordings while our lab used 31 channel recordings. Although several labs have utilized additional channels to sample the EEG, it is generally accepted that the spatial EEG acquired by commonly used systems is undersampled. With this knowledge in hand, it is then rather surprising that spatial analysis methods, such as topographic mapping [1], are routinely performed on this undersampled data. One should obviously question the validity of the results obtained from the spatial analysis of undersampled data.

It is well-known that the effect of undersampling can be described as the aliasing of spectral components. In order to quantify the magnitude of the aliased component, the spatially aliased proportion of the signal's variance is estimated, which requires knowledge of the signal's spatial power spectral density function (PSDF). The purposes of this chapter are to estimate the spatial PSDF of the EEG, to estimate the spatially aliased proportion of the signal's variance for a 31-electrode montage and to estimate the temporal PSDF of the aliased component.

5.1 Introduction

Estimating the spatially aliased portion of the signal's variance is directly related to the problem of determining the minimal number of electrodes required to adequately sample the spatial EEG. Since the spatial extent of the sampling is fixed to the upper half of the scalp, increasing the number of electrodes decreases the spatial sampling interval. The following two approaches have been used to determine the minimum number of electrodes.

One approach to determine the minimal number of electrodes is based on the estimated point spread function of a current dipole generator. A current dipole is assumed to be located on the outer layer of the cerebral cortex and the resulting potential distribution on the scalp due to volume conduction theoretically determined. The required number of electrodes can then be defined as that number that ensures that the largest measured potential is greater than some percentage, say 50%, of the peak value due to any dipole source on the cerebral cortex. Using this approach, it has been estimated that approximately 200 electrodes are required to adequately sample the spatial EEG, although this estimate has yet to be experimentally confirmed.

An alternate approach to determining an adequate number of electrodes is based on sampling theory. The Nyquist theorem [2] states that the minimum sampling rate, referred to as the Nyquist rate, of an adequately sampled signal is twice the highest wavenumber component in the signal. If the sampling rate is less than the Nyquist rate, the signal's higher wavenumber components will alias. It is possible to estimate the aliased portion of the undersampled signal's variance given knowledge of the signal's spatial PSDF. For a given sampling rate, the aliased portion of the signal will be those wavenumber components that are larger than one-half the sampling rate. Since the spatial spectrum of the EEG is not bandlimited, one can then specify that an adequate sampling rate results in the aliased portion of the signal's variance being less than some acceptable level.

Spectral analysis methods based on the discrete Fourier transform (DFT) can be used to estimate the sampled signal's spatial PSDF. Unfortunately, the sampled EEG is not properly spatially bandlimited, as discussed in chapter 2, which results in the sampled signal's estimated spatial PSDF being an aliased version of the continuous signal's spectrum.

In order to reduce the aliasing in the estimated spatial PSDF, the sampling rate can be increased. It is usually not practical to increase the sampling rate in both dimensions for a two-dimensional (2-D) signal due to the quadratic increase in the number of required samples. Instead, a simpler approach is to increase the sampling rate along a one-dimensional (1-D) path. Using this approach, both linear and circular arrays of closely spaced electrodes have been used to sample the EEG of human [3] [4] and animal subjects [5]. A similar approach has also been used to estimate the spatial spectrum of the earth's magnetic field [6] by sampling at equidistant points on a great circle track. The major assumption with this approach is that the signal's 2-D spatial PSDF has radial symmetry about the origin.

5.2 Data Collection and Preprocessing

It is desired to investigate the aliased component of the signal's variance due to use of the normal 31-channel montage shown in Figure 5.1. The normal montage spatially samples the EEG on an approximately rectangular grid, with a 5 cm. average interelectrode distance. In order to increase the spatial sampling rate, a circular montage was constructed consisting of 31 equally spaced electrodes, also shown in Figure 5.1. Since the circumference of the circular montage was approximately 60 cm, the average interelectrode distance was on the order of 2 cm. Note that it would take approximately 190 electrodes evenly distributed over the scalp to have an average interelectrode distance of 2 cm.

Four subjects' EEGs were recorded, for the eyes positioned (EP) and eyes closed (EC) conditions, at a rate of 120 samples/sec using the circular montage. The reference electrode was placed in the middle of the circular montage, as shown in Figure 5.1. The best 20-second epoch was selected from each record on the basis of containing minimal artifacts, which was then phase and magnitude corrected, as described in chapter 2.

In order to eliminate the effects of the active reference on the recorded signal, the spatial average was removed from the data as follows.

$$u_i(n) = v_i(n) - \frac{1}{31} \sum_{j=0}^{30} v_j(n) \quad (5.2.1)$$

where $v_i(n)$ is the measured potential of channel i for temporal sample n and $u_i(n)$ is the reference free potential. Due to the fact that all of the electrodes are perimeter electrodes, (5.2.1) can be considered as either the average or perimeter referenced signal, as described in chapter 4. The linear relationship between $v_i(n)$ and $u_i(n)$ described by (5.2.1) can be considered as high-pass spatial filtering operation. However, due to the fact that the data was circularly sampled, the characteristics of the spatial filter are shift invariant and the previously outlined problems for the average reference method avoided.

As a preliminary data analysis step, composite temporal PSDF estimates were obtained from the average referenced epochs, with the results shown in Figure 5.1. These composite spectra were calculated by averaging the temporal PSDF estimates of each channel which, in turn, were determined via Welch's method [7]. Welch's method of spectral estimation was applied to the EEG epochs by dividing each channel's data into 128 point Hanning windowed segments with 50% overlap. The magnitude squared DFT values of each segment and channel were averaged together, then normalized by the sum of the squares of the data, resulting in the composite PSDF estimate.

The main difference between the EC and EP composite PSDF estimates, shown in Figure 5.1, is a peak centered at approximately 10 Hz. in the EC spectra. It is well-known that EEGs recorded under the EC condition often contain alpha activity, which occurs in the 8-12 Hz. frequency band. Alpha activity is then suppressed when the subject's eyes were opened.

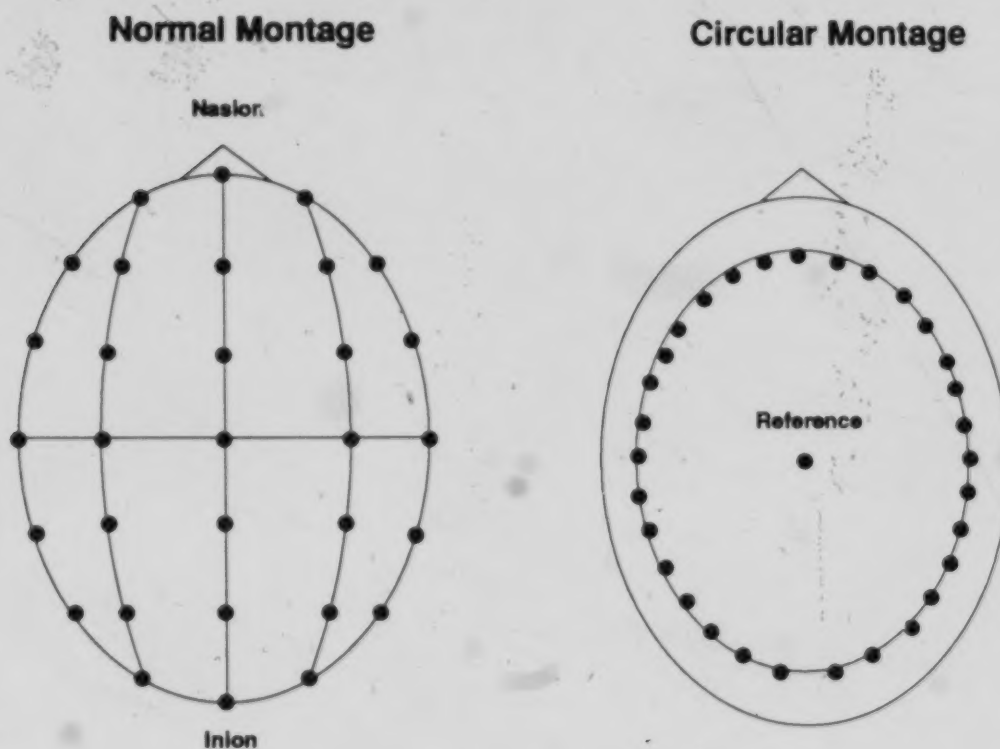
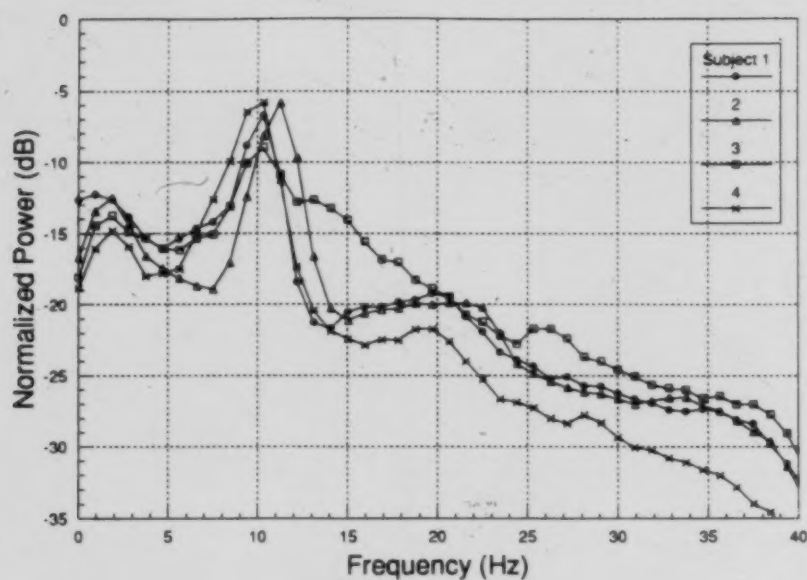
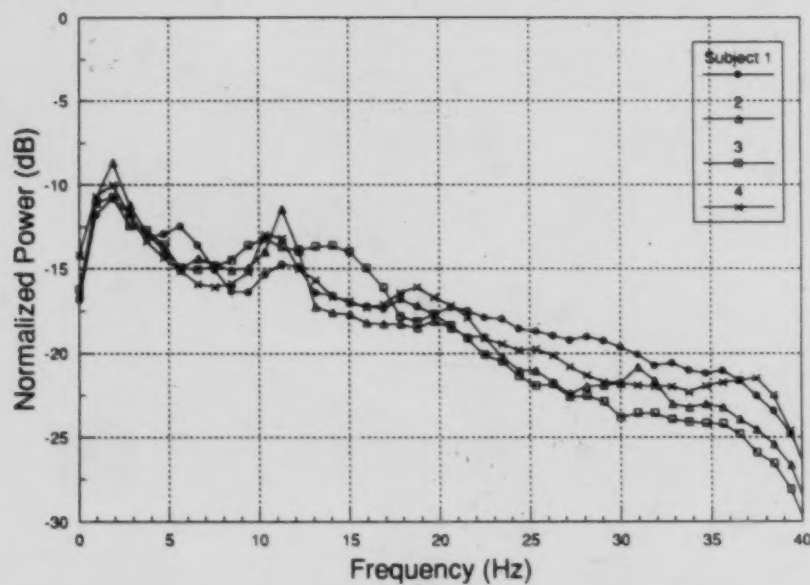


Figure 5.1 - Normal and circular 31-electrode montages used to sample the EEG. The circumference of the circular array of electrodes was approximately 60 cm, which corresponds to a sampling interval of approximately 2 cm while the normal montage has a 5 cm average distance between adjacent electrodes.



(a) Composite temporal spectra for the EC epochs



(b) Composite temporal spectra for the EP epochs

Figure 5.2 - Composite temporal PSDF estimates for the EC and EP average referenced epochs. Note the peak in the EC spectra at approximately 10 Hz. due to alpha activity.

5.3 Analysis Methods and Results

5.3.1 Spatial Spectral Analysis

The spatial spectrum of each average referenced epoch was estimated via the averaged periodogram method [8]. Assuming that the spatial signal was stationary during the 20 second recording, the averaged periodogram method proceeds by averaging the magnitude squared values of the spatial DFTs as follows

$$P(k) = \frac{1}{2400} \sum_{n=0}^{2399} \left| \sum_{i=0}^{30} u_i(n) \exp(-j2\pi ik/31) \right|^2 \quad 0 \leq k \leq 30 \quad (5.3.1)$$

where k will be referred to as the wavenumber index. Note that a spatial window function in the above equation is not required due to the implicit periodicity of the circular sampling. Also note that for each epoch, the zero wavenumber term $P(0)$ equals zero due to the use of the average reference. The proportion of the signal's variance at wavenumber indices of $\pm k$ is given by

$$P_n(k) = \frac{P(k)}{\sum_{i=1}^{15} P(i)} \quad (5.3.2)$$

where $P_n(k)$ will be referred to as the normalized spatial PSDF estimate.

Given the fact that the circular montage's circumference averaged approximately 60 cm, k corresponds to a wavenumber of $k/60$ cycles/cm. Since the circumference of the human head is approximately 60 cm, k also signifies the number of cycles in a spatial wave that encompasses the head. For simplicity, k will be used in plotting the spatial spectra instead of its associated wavenumber. Note that the highest wavenumber that the 31-electrode circular montage can detect in the spatial EEG is 0.25 cycles/cm, which corresponds to a wavenumber index of $k = 15$.

Shown in Figure 5.3 is a log-log plot of $P_n(k)$ vs. k for the selected EEG epochs. From these plots, it can be noted that there exists an approximately linear relationship between the log spectral power and the smaller log wavenumber indices. For larger wavenumber indices, the rolloff rate of the spectrums appear to be reduced which could be the result of several factors. The most obvious factor in reducing the rolloff rate is the spatial aliasing due to sampling. Assuming that the actual spectrum of the signal monotonically decreases with increasing values of k , aliasing will most significantly increase the power of those spectral components that are situated near the folding frequency (half the sampling rate). Additional factors for reducing the rolloff rate include the effects of quantization and measurement noise and the slight variations in the positions in the electrodes. For these reasons, it is concluded that the most reliable spectral estimates are obtained for the lower wavenumber indices.

For a given spatial sampling rate, the normalized aliased variance E will be defined as the spatially aliased component of the signal's variance normalized by the original signal's variance. The main objective of this chapter is to estimate E for the regular 31-electrode

montage shown in Figure 5.1, which has an average sampling interval of 5 cm. Two methods of estimating E will be proposed: one method that assumes that the original signal is spatially bandlimited and another method that is based on an extended spatial spectrum.

One approach to estimating E is to assume that the EEG was spatially bandlimited to wavenumbers less than 0.25 cycle/cm, which infers that $P_n(k)$ obtained from the circular montage do not contain aliased components. With this assumption, an estimate of E based on the circular montages PSDF estimates is given by

$$E_1(K) = \sum_{k=K+1}^{15} P_n(k) \quad (5.3.3)$$

where K will be referred to as the folding wavenumber index and where the value for K is determined by the average interelectrode distance. For the normal montage, aliasing will occur for wavelengths smaller than 10 cm, because the distance between adjacent electrodes is on the order of 5 cm. Since a wavelength of 10 cm relates to a folding wavenumber index of $K = 6$, $E_1(6)$ gives an estimate of E for a 5 cm sampling interval. A plot of $E_1(K)$ vs. K is shown in Figure 5.4.

Note that since the actual EEG signal was not spatially bandlimited, the circular montages sampled signal will be aliased which will affect the estimates of $E_1(K)$. The effect of aliasing can be minimized by restricting the use of $E_1(K)$ to low values of K . As explained previously, the spectral values most affected by aliasing should be located at approximately half the sampling rate. Thus, as long as the sampling rate of the circular montage is much higher than the sampling rate of the regular montage, the effect of aliasing on $E_1(K)$ should be minimized.

An alternative method of estimating E is to use a model to extend the spectral estimates. The spectral estimates shown in Figure 5.3 indicate an exponential relationship between the normalized power $P_n(k)$ and the wavenumber index k as follows

$$P_n(k) = Ck^{-\alpha} \quad k > 0 \quad (5.3.4)$$

where C represents a scale factor and where α signifies the roll off rate of the spatial PSDF. The above model for the spatial PSDF corresponds to a roll off rate of 3α dB/octave. It can be shown that the parameter α is independent of both the reference level and the normalization factor used to compute $P_n(k)$. Linear regression was used to determine the value of the parameter α from each epoch's log-log spatial spectrum over the range $1 \leq k \leq 6$, with the results shown in Table 5.1. Estimates of the scale factor C are not tabulated, for this parameter does convey any useful information.

	EC Data	EP Data
Subject 1	2.72 $\pm$ 0.05	2.04 $\pm$ 0.05
Subject 2	3.07 $\pm$ 0.06	2.55 $\pm$ 0.11
Subject 3	2.92 $\pm$ 0.06	2.62 $\pm$ 0.05
Subject 4	3.13 $\pm$ 0.27	2.07 $\pm$ 0.10
Average	2.96	2.32

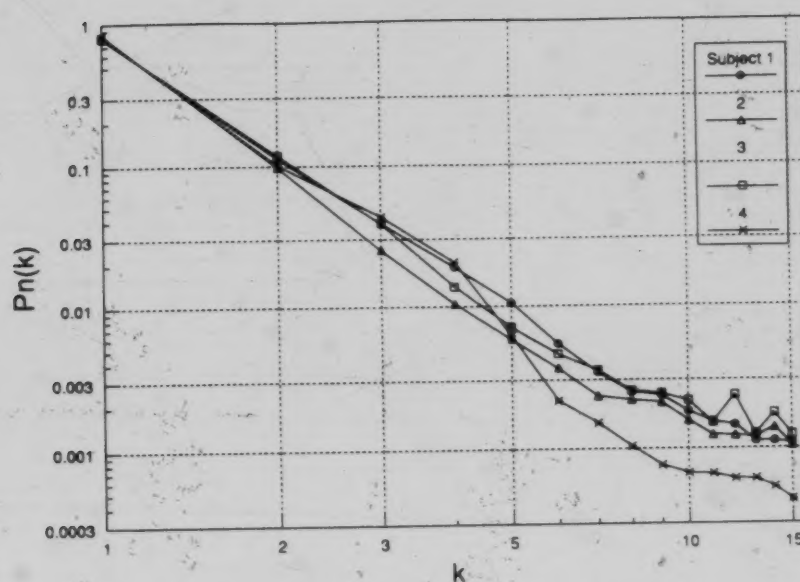
Table 5.1 - Estimates for α based on the log-log spatial spectral plots shown in Figure 5.3 and the model $P_n(k) = Ck^{-\alpha}$ fitted over the range $1 \leq k \leq 6$. Note that the EC epochs have a larger value for α , which infers that its spatial bandwidth is smaller.

Assuming that the exponential relationship given by (5.3.4) is valid for $k > 0$, an estimate of E based on the extended spectra is given by

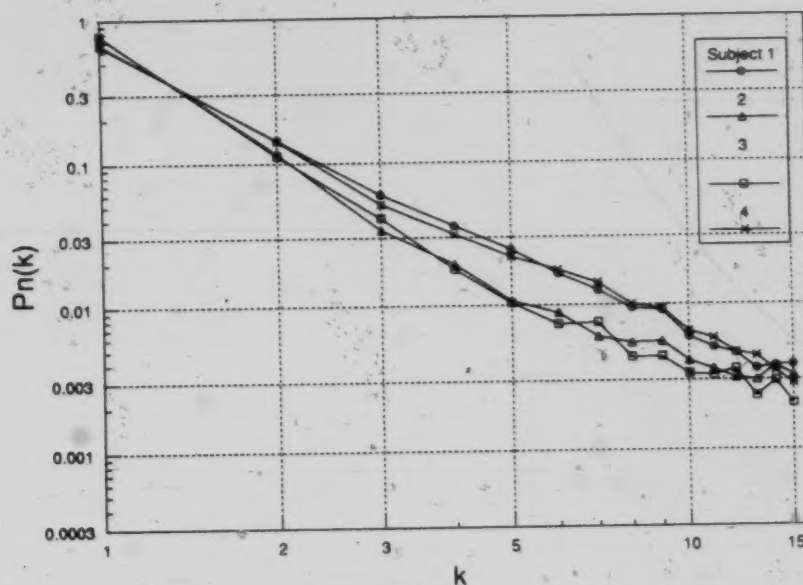
$$E_2(K, \alpha) = \frac{\sum_{k=K+1}^{\infty} k^{-\alpha}}{\sum_{k=1}^{\infty} k^{-\alpha}} \quad (5.3.5)$$

$$= \frac{\zeta(\alpha) - \sum_{k=1}^K k^{-\alpha}}{\zeta(\alpha)}$$

where $\zeta(\alpha)$ is the Riemann zeta function [9]. Note that $E_2(K, \alpha)$ is reference free due the exclusion of the zero wavenumber term $k = 0$ in (5.3.5). Shown in Figure 5.5 is a plot of $E_2(6, \alpha)$ vs. α which corresponds to a sampling interval of 5 cm.. In order to test the validity of the exponential model for extending the spectra, $E_1(6)$ was also plotted vs. the estimated α obtained from each of the epochs. As it can be observed from Figure 5.5, there exists a reasonable correspondence between $E_2(6, \alpha)$ and the $E_1(6)$ values.

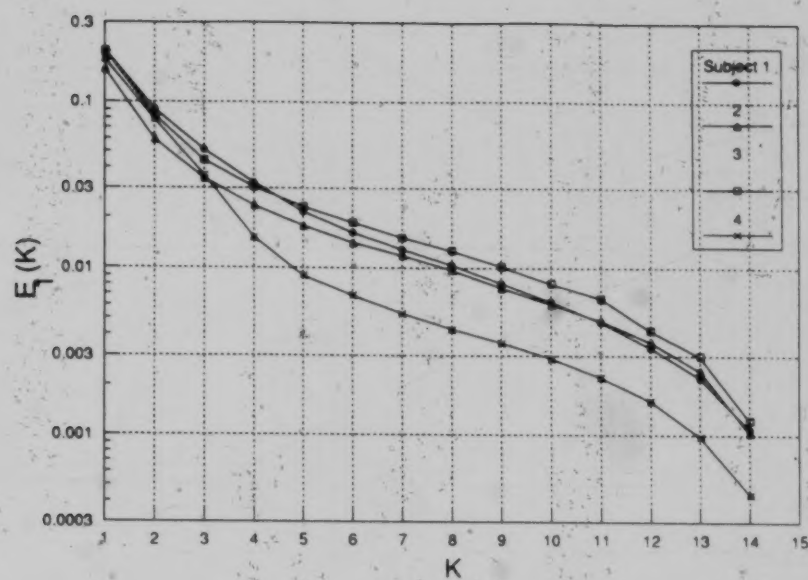


(a) Spatial spectra for the EC epochs

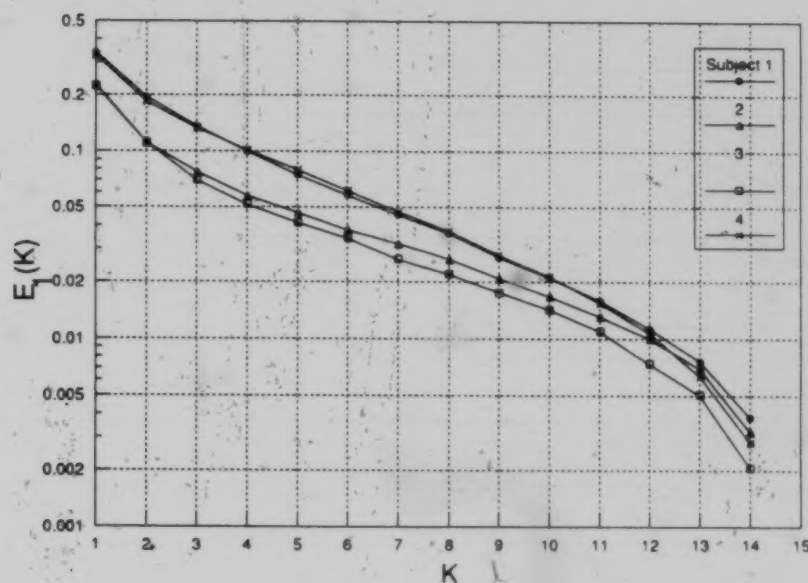


(b) Spatial spectra for the EP epochs

Figure 5.3 - Normalized spatial PSDF estimates of four subjects' average referenced EEGs recorded with the circular montage. Note the higher roll off rate for the EC spectra when compared to the EP spectra. Also, both sets of spectra exhibit a slight flattening at higher wavenumbers, probably due to the effects of additive noise and aliasing.



(a) Aliased variance for the EC epochs



(b) Aliased variance for the EP epochs

Figure 5.4 - $E_1(K)$ vs. the folding wavenumber index K for the average referenced epochs. These plots illustrate the portion of the signal's variance that would be spatially aliased for an average sampling interval of $30/K$ cm. The most significant results relate to $K = 6$ which corresponds to the 5 cm. sampling interval of the normal montage.

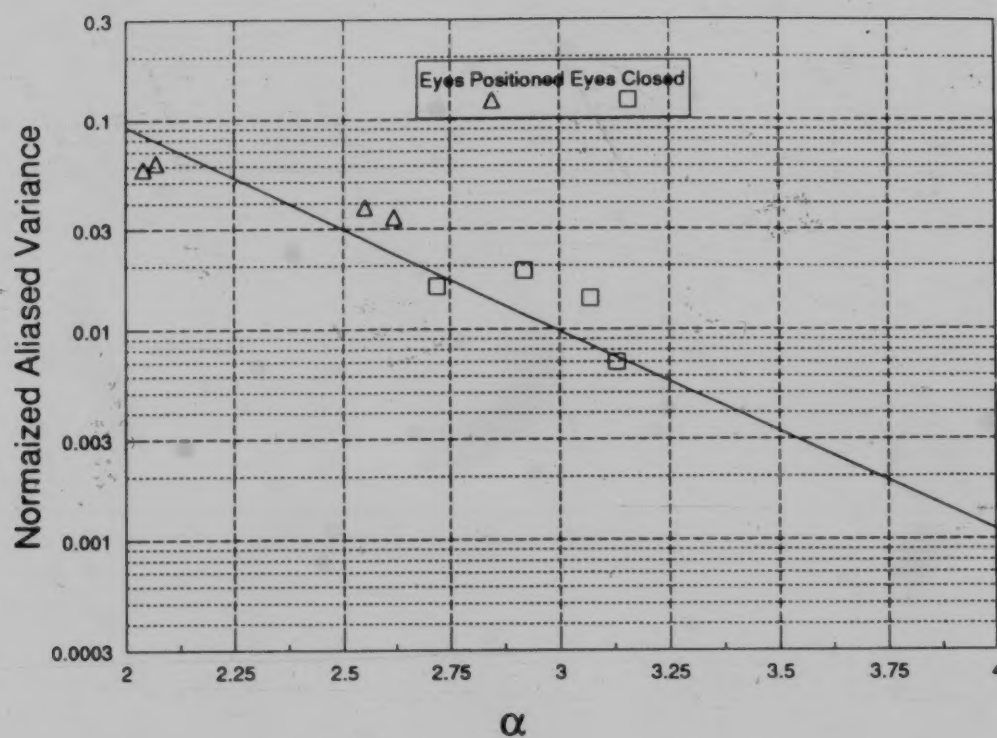


Figure 5.5 - Plot of $E_A(6, \alpha)$ and $E_I(6)$ vs. α . The function $E_A(6, \alpha)$, plotted as a solid line, estimates the spatially aliased proportion of the EEGs variance for a sampling interval of 5 cm. as a function of α . The triangles and squares indicate the estimated aliased variance based on the $E_I(6)$ values given in Figure 5.4 and the associated parameter α shown in Table 5.1.

5.3.2 Temporal Spectral Analysis of the Aliased Signal

This section estimates the temporal spectrum of the spatially aliased component of the EEG via a frequency-wavenumber spectral analysis. The results obtained from this section also provides a measure of the signal's spatial bandwidth as a function of temporal frequency.

The frequency-wavenumber spectrum of each average referenced epoch was estimated using Welch's method extended to 2-D data [10]. The 2-D DFT of the m -th windowed section of the average referenced epochs was calculated by

$$U_m(k, w) = \sum_{i=0}^{30} \sum_{n=0}^{127} y(n) u_i(n + 64m) \exp\left(-j \frac{2\pi}{31} ki - j \frac{2\pi}{128} wn\right) \quad (5.3.6)$$

$$0 \leq k \leq 30 \quad 0 \leq w \leq 127 \quad 0 \leq m \leq 39$$

where k is the wavenumber index, w is the frequency index and $y(n)$ is a window function of the following form

$$y(n) = \frac{1}{2} \left(1 - \cos\left(\frac{2\pi n}{128}\right) \right) \quad (5.3.7)$$

Note that the window function has a constant spatial value, again taking advantage of the implicit periodicity in the circular spatial sampling. The index w corresponds to a temporal frequency of $120w/128$ Hz. An estimate of the signal's 2-D PSDF was obtained by averaging the magnitude-squared 2D-DFT values as follows

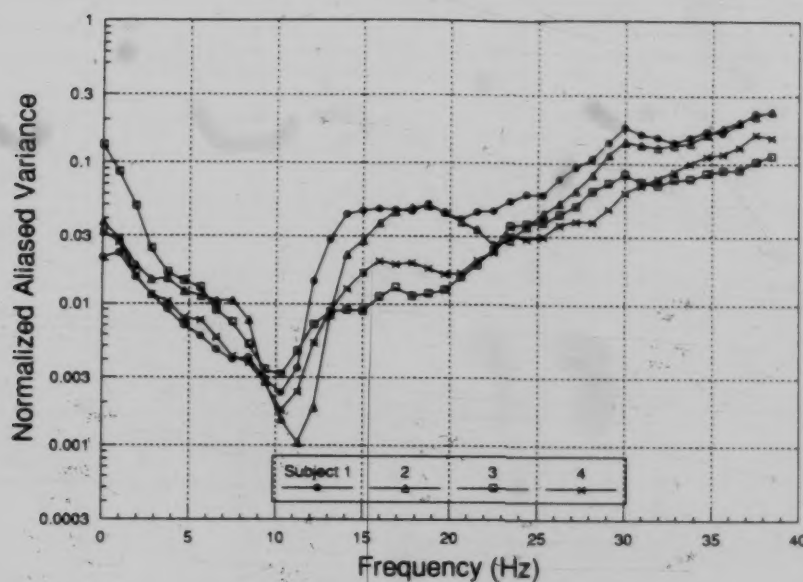
$$P(k, w) = \frac{1}{40} \sum_{m=0}^{39} U_m(k, w) U_m^*(k, w) \quad (5.3.8)$$

where $*$ is the complex conjugate operator.

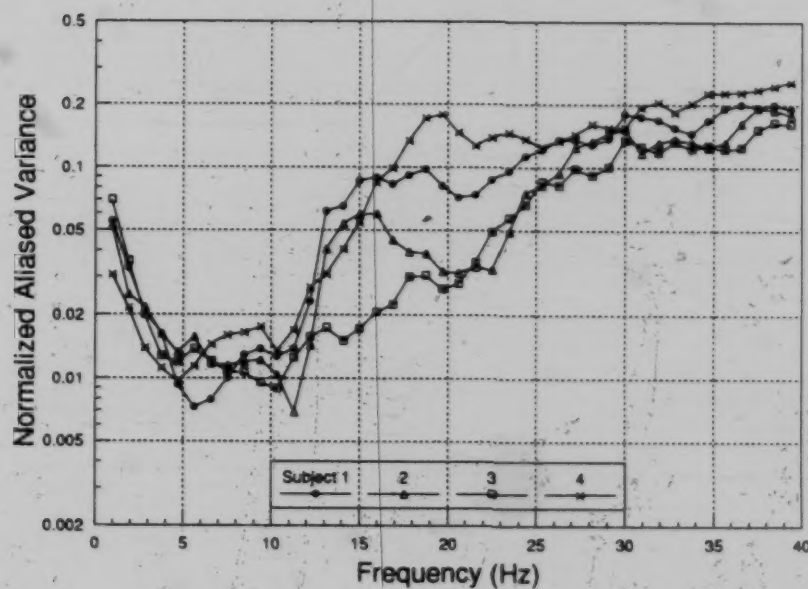
An estimate of E as a function of frequency can be defined by

$$E_3(K, w) = \frac{\sum_{k=K+1}^{15} P(k, w)}{\sum_{k=1}^{15} P(k, w)} \quad (5.3.9)$$

where K is the folding wavenumber index of the electrode montage. Note that the definition for $E_3(K, w)$ assumes that the signal sampled by the circular array was not aliased, which was the same assumption made in the definition for $E_1(K)$. Again, by restricting the use of $E_3(K, w)$ to small values of K , the effects of aliasing in the signal sampled by the circular array can be minimized. The function $E_3(K, w)$ can also be used as a measure of the spatial bandwidth of the signal as a function of frequency, with the smallest spatial bandwidth occurring at those temporal frequencies where $E_3(K, w)$ is smallest. Shown in Figure 5.6 is a plot of $E_3(6, w)$ vs. frequency, which corresponds to an estimate of the spatially aliased proportion of the variance for a 5 cm sampling interval.



(a) EC



(b) EP

Figure 5.6 - Temporal spectral plots of $E_3(6, \omega)$ for the average referenced epochs. These plots estimate the normalized spatially aliased variance as a function of frequency for a sampling interval of 5 cm. Note that in both sets of plots, the aliased variance is minimal in the alpha range (8-12 Hz.) of frequencies, indicating that the signal's spatial bandwidth is smallest in that range.

5.4 Discussion of Results

Visual inspection of the spatial EEG PSDF estimates suggested the exponential model $P(k) = Ck^{-\alpha}$. Estimates of α , obtained from linear regression, ranged from 2 to 3 which corresponds to a spatial PSDF roll off rate of 6 to 9 dB/octave for the EP and EC epochs respectively. These results are consistent with the > 6 dB/octave spectral roll off rate obtained by Gevins [4] for scalp recorded visual event-related potentials. It can also be noted that for each subject, the estimated value for α was larger for the EC epoch than for the EP epoch. This infers that the spatial bandwidth of each subject's EEG was smaller for the EC epoch.

The spatial roll off rate of 6-9 dB/octave for EEG raises a serious question about the validity of the spatial Laplacian maps estimated from the EEG. As was shown in chapter 3, the spatial Laplacian is proportional to the second spatial derivative of the EEG which causes the spatial spectrum to be increased by +12 dB/octave. Using this relationship, one could predict that the spatial PSDF of the radial current estimate should actually increase with increasing wavenumbers! For a given sampling interval, the spatially aliased component of the signal's variance would be larger than the unaliased component. However, this problem could be reduced by spatially bandlimiting the spatial Laplacian operator.

Two methods were used to estimate the spatially aliased portion of the EEG's variance for a sampling interval of 5 cm. With the assumption that the EEG was adequately spatially sampled by the circular montage, the estimated normalized aliased variance results obtained from Figure 5.4, $E_1(6)$ ranged from 0.007-0.019 and 0.034-0.062 for the EC and EP epochs respectively. The normalized aliased variance estimates based on the exponentially extended spectra given in Figure 5.5, $E_2(6, \alpha)$ ranged from 0.0098-0.093 as α ranged from 3 to 2. These results indicate that the EC epochs have significantly less aliased variance than the EP epochs, again inferring that former's spatial bandwidth is smaller.

The next major set of results involved estimating the normalized temporal spectra of the spatially aliased component of the signal for a sampling interval of 5 cm. From the $E_3(6, w)$ results shown in Figure 5.6, the following points can be noted.

1. The aliased variance tends to increase as the frequency increases beyond approximately 11 Hz. One possible explanation for this effect is the existence of wave activity in the EEG, which infers a relationship between spatial wavenumbers and temporal frequencies. It has been postulated that the brain's electrical activity is a result of waves forming on the outer layer of the cerebral cortex as discussed in Chapter 2. It has also been proposed by Nunez [1] that there exists a relationship between the wavenumber and frequency content, with higher frequency components being associated with higher wavenumbers. This relationship infers that the spatial bandwidth of the signal, along with the sampled signal's aliased variance, increases with increasing frequencies. Another contributing factor to this effect is that the differences in the channels' frequency responses also increase with increasing frequencies which will tend to increase the spatial bandwidth of the signal. However, this effect should be minimal due to the magnitude and phase correction of each channel.
2. As the frequency approaches zero, the aliased variance increases. One possible cause for this effect is the increasing phase differences between the channels as the lower cutoff frequency of the analog band-pass filters is approached. Note that the channel's low frequency responses were not corrected.

3. For the EC epochs, a sharp minimum in the aliased variance occurs in the alpha frequency range 8-11 Hz. Another prediction of the brain wave theory presented by Nunez is that alpha activity could be the result of a spatial standing wave, with a wavelength equal to the head's circumference forming on the outer layer of the cerebral cortex. This infers that the spatial bandwidth of the alpha activity should be small when compared to the spatial bandwidth of activity in other frequency bands. This predicted small spatial bandwidth of the alpha activity is supported by the sharp reduction in the aliased variance in the alpha band of the EC epochs. Visual inspection of both the EEG epochs and the composite PSDF spectral plots shown in Figure 5.2 confirmed the presence of alpha activity in these epochs.

4. For the EP epochs, a minimum in the aliased variance occurs in the frequency range 4-11 Hz, although not as significant as for the EC epochs. One possible explanation of this effect is that alpha activity is still present, although at a reduced level, in the EP epochs. Inspection of the EP EEG epochs and the composite PSDF estimates shown in Figure 5.2 indicated a reduction in the alpha activity when compared to the EC epochs.

As a final note, since the experimental results generally support the existence of a dispersive relationship in the EEG and the sharp reduction in the spatial bandwidth for alpha activity, they also support the existence of brain waves, as described by Nunez. Further experimental support for the existence of brain waves will be presented in following chapters.

5.5 Conclusions

The experimental results based on four subjects' spatial PSDF estimates indicate that the spectrum of the background EEG has a roll off rate ranging from approximately 6 to 9 dB/octave, with the EC spectrum having a faster roll off rate than the EP spectrum. It has also been estimated that the spatially aliased portion of the signal's variance due to the use of a 31-electrode montage ranges approximately from 0.001-0.08, again with the EC epochs having the smallest aliased variance. The results also indicated that the alpha frequency band had the smallest amount of spatial aliasing.

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6 Comparison of Several Spatial Models for the EEG

Representing the spatial EEG by a mathematical model has several applications, namely: to spatially smooth the EEG, to extract the spatial trend from the EEG, to interpolate the EEG into a format that is suitable for viewing purposes (topographic mapping) and to localize abnormal activity in the EEG. Several of these applications will be considered in this and the following chapters.

A smoothing or an interpolating model can be used to describe the spatial EEG. These two approaches to modeling differ in their basic assumption about the quality of the observed data. Interpolation models consider the experimental data as noise free, and result in a fitted function that matches the experimental data. Smoothing models, in contrast, consider the experimental data as a sum of a predictable and an unpredictable (noise) component. The fitted function attempts to match only the predictable portion of the signal while ignoring the unpredictable component, resulting a function that only approximates the data.

This chapter compares the adequacy of the bilinear, spherical spline and truncated spherical harmonic expansion models in describing the spatial EEG using the method of cross-validation. In order to establish the suitability these models, the optimal linear model was obtained via multiple linear regression (MLR). Finally, the optimal estimation of several model specific parameters was considered.

6.1 Cross-Validated Models

In order to determine the merit of a model in describing the spatial EEG, one might be tempted to fit the model to the data, then calculate the variance of the error (residual) signal. A small residual variance would then indicate a good model fit. Unfortunately, an interpolating model has zero error due to the matching of the observed data by the fitted function. A smoothing model solved via a least squares approach produces a residual variance that will decrease as the number of degrees of freedom in the model increases, and overfitting can result.

A better test of the adequacy of the model can be performed by testing the predictability of the model. The data can be split into two groups, with one group used to form the model and the second group used to test the suitability of the model. For example, the 31-channel EEG data could be supplemented with additional channels. By fitting the model to the 31-channel data and testing it on the additional channels, the problems mentioned in the previous paragraph can be minimized. Unfortunately, it was not possible to take additional spatial samples of the EEG due to hardware constraints.

In order to provide two separate data groups for fitting and testing the model, a simpler approach is to split the 31-channel data set into two smaller sets of say, 21 and 10 channels. However, splitting the original data set into two groups has the problem that the data set used to form the model is smaller than the original data set, which results in a less accurate model. The optimal model for the smaller data set might not be the optimal model for the larger data set.

A better approach to testing the adequacy of a model is to use cross-validation [1]. Cross-validating a model involves removing one variable from the data, fitting the model to the remaining variables, then using the model to predict the removed variable. Specifically, cross-validation involves the following steps.

1. Remove spatial sample (channel) i from the data. This is equivalent to removing column i from the data matrix $\bar{V}$.
2. Solve for the model's coefficients.
3. Using the model obtained in step 3, and the spatial location of sample i , predict the activity of sample i .
4. Repeat the above steps for all the spatial samples.
5. Determine the adequacy of the prediction by finding the variance of the residual signal.

The adequacy of the model can be tested by calculating several statistical parameters, as described in a following section.

6.2 Spatial Models

The following three cross-validated spatial models of the EEG will be considered in this section: the bilinear, spherical spline and spherical harmonic expansion models. The interpolating bilinear model is the easiest model to implement, but often produces poor results when compared to more involved models. The interpolating or smoothing spherical spline model is derived from a set of equations that are common to a large class of spline functions. The truncated spherical harmonic expansion model can be used to spatially smooth the data along with extracting the spatial trend for the data.

6.2.1 Bilinear Model

One of the simplest interpolation methods for a 2-D signal sampled on a rectangular grid is to use a bilinear model. The sampled region is first divided into a set of adjacent rectangular regions, with each corner of the rectangular regions corresponding to a sampling point. A bilinear model, given by

$$\hat{v}(x, y) = a + bx + cy + dxy \quad (6.2.1)$$

is then fitted to each region, where a, b, c, d are chosen to match the observed data at the four corner points of a rectangular region, x, y represent the rectangular coordinates of the region and $\hat{v}(x, y)$ is the predicted potential. This method leads to a continuous interpolating function with discontinuous first-derivatives at the boundaries between the rectangular regions.

Assuming that the signal is sampled on a rectangular grid, it can be shown that the bilinear estimate of the activity of spatial sample (i, j) is given by

$$\hat{v}(i, j) = \frac{1}{4} [v(i+1, j) + v(i-1, j) + v(i, j+1) + v(i, j-1)] \quad (6.2.2)$$

Equation (6.2.2) predicts the activity of spatial sample (i, j) based on the average activity of the four nearest neighbors. Note that (6.2.2) can only be used to predict the activity of samples located inside the sampling grid.

Samples located on the perimeter of the sampling grid can be predicted by using linear interpolation. For convenience, assume that the perimeter samples are circularly indexed. The predicted activity of perimeter sample k is obtained by taking the average of the two adjacent perimeter samples, as given by

$$\hat{v}(k) = \frac{1}{2} [v(k+1) + v(k-1)] \quad (6.2.3)$$

Assuming that the sampling grid covers a rectangular region, the four corner samples cannot be reliably predicted by a bilinear model. Fortunately, the perimeter samples of the EEG forms an approximately circular path and thus contains no corner samples.

It can be noted that the residual signal produced by the bilinear model for interior samples, given by

$$e(i, j) = v(i, j) - \frac{1}{4} [v(i+1, j) + v(i-1, j) + v(i, j+1) + v(i, j-1)] \quad (6.2.4)$$

provides an scaled estimate of the spatial Laplacian [2]. As was mentioned in chapter 3, the spatial Laplacian of the potential function provides an estimate of the radial current entering and exiting the scalp and is used for localization.

6.2.2 Spherical Spline Model

Spherical splines [3] [4] [5] have been proposed as either smoothing or interpolating functions for spherically sampled data. A spherical spline can be considered as a thin, flexible spherical shell that is deformed to approximate the data in an optimal fashion. In general, the concept of a spline is to minimize a measure of the magnitude of the κ -th spatial derivative in the fitted function, resulting in a well-behaved function. One measure of the magnitude of the κ -th spatial derivative a spherical function v is given by

$$J_{\kappa}(v) = \int_S (\Delta^{\kappa} v)^2 ds \quad (6.2.5)$$

where Δ is the spherical Laplace-Beltrami operator, S represents the surface of the sphere and κ is a positive integer > 1 . The parameter J_{κ} represents the bending energy of a thin spherical shell when $\kappa = 2$. Increasing values of κ results in J_{κ} measuring higher spatial derivatives in the fitted function. J_{κ} can also be calculated from the spherical harmonic expansion of the function v by

$$J_{\kappa}(v) = \sum_{n=1}^{\infty} \sum_{m=-n}^n (a_n^m)^2 [n(n+1)]^{\kappa} \quad (6.2.6)$$

where a_n^m are the spherical harmonic coefficients.

Consider a potential function that is sampled on the surface of a sphere over time. Let $v_i(n)$ represent the potential of spatial sample (channel) i and temporal sample n , where $1 \leq i \leq N_s$ and $1 \leq n \leq N_t$. Given a spatial data set $\{v_i(n), 1 \leq i \leq N_s\}$ where n is constant, the following two problems were outlined and solved by Wahba.

1. Smoothing ($\lambda > 0$)

$$\text{minimize } \frac{1}{N_s} \sum_{i=1}^{N_s} (v_i(n) - \hat{v}_i(n))^2 + \lambda J_s(\hat{v}) \quad (6.2.7a)$$

2. Interpolation ($\lambda = 0$)

$$\text{minimize } J_s(\hat{v}) \quad \text{subject to } v_i(n) = \hat{v}_i(n) \quad i = 1, 2, \dots, N_s \quad (6.2.7b)$$

where $\hat{v}$ is the spherical spline estimate of the function v . The first term in (6.2.7a) represents the squared error between the estimated and actual data while the second term indicates the roughness of the estimated function. Increasing values of λ increases the significance of the roughness term, resulting in smoother estimates. When $\lambda = 0$, the fitted surface matches the given data, resulting in an interpolating function.

It can be shown that the solution to both (6.2.7a) and (6.2.7b) can be written as

$$\hat{v}(P) = \sum_{i=1}^{N_s} c_i q_s(\cos(P, P_i)) + d \quad (6.2.8)$$

where

$$q_s(x) = \frac{1}{4\pi} \sum_{n=1}^{\infty} \frac{2n+1}{n^2(n+1)^2} P_n(x),$$

P_i represents the spherical coordinates of channel i , (P, P_i) is the angle between the points P and P_i and $P_n(x)$ is the n -th degree Legendre polynomial. In order to numerically evaluate the above expression for $q_s(x)$, the infinite summation can be truncated at a suitably high degree (~ 50). The coefficients $\{c_i, 1 \leq i \leq N_s\}$ and scalar d in (6.2.8) can be determined with the aid of the following two equations

$$(\bar{Q}_s + N_s \lambda \bar{I}) \bar{c} + d \bar{t} = \bar{v}(n) \quad (6.2.9a)$$

$$\bar{t}^T \bar{c} = 0 \quad (6.2.9b)$$

where

$$\bar{c} = [c_1, c_2, \dots, c_{N_s}]^T,$$

$$\bar{t} = [1, 1, \dots, 1]^T \quad (\text{size } N_s),$$

$$\bar{v}(n) = [v_1(n), v_2(n), \dots, v_{N_s}(n)]^T,$$

$\bar{Q}_s$ is a N_s by N_s sized matrix with elements defined by $(Q_s)_{ij} = q_s(\cos(P_i, P_j))$, T is the vector transpose operator and $\bar{I}$ is a N_s by N_s sized identity matrix.

In order to predict the activity of sample i , the following procedure was followed.

1. Sample i was deleted from the data vector $\vec{v}(n)$, and the parameters $\vec{c}$ and d determined from equations (6.2.9a) and (6.2.9b).
2. Equation (6.2.8) was then used to predict the activity of the deleted sample i .

It should be noted that (6.2.8) and (6.2.9) do not need to be explicitly solved for each set of spatial observations. Instead, both equations can be combined into one matrix equation with the weighting factors dependent only on the positions of the channels. This approach allows efficient evaluation of the spherical spline function.

One problem encountered with using the spherical spline model is the determination of the optimal values for the parameters κ and λ . Again, the method of cross validation can be used to determine the parameters values that result in the smallest model error for a given data set.

6.2.3 Truncated Spherical Harmonic Expansion

As discussed in chapter 3, the degree N spherical harmonic expansion of the sampled signal is given by $\vec{Y}\vec{a} \approx \vec{v}(n)$ where $\vec{Y}$ is a N_s by $(N+1)^2$ sized matrix consisting of the sampled spherical harmonic functions, and $\vec{a} = [\vec{Y}^T \vec{Y}]^{-1} \vec{Y}^T \vec{v}(n)$ is a N_s sized coefficient vector. The predicted activity at a point P located on the surface of the sphere is given by

$$\hat{v}(P) = \sum_{n=0}^N \sum_{m=-n}^n a_n^m Y_n^m(P) \quad (6.2.10)$$

$$= \vec{y}^T(P) [\vec{Y}^T \vec{Y}]^{-1} \vec{Y}^T \vec{v}(n)$$

where

$$\vec{y}(P) = [Y_0^0(P), Y_0^1(P), \dots, Y_N^N(P)]^T$$

is a $(N+1)^2$ sized vector. It is assumed that the number of spherical harmonic coefficients is less than the number of data points, which results in a smoothing function.

The predicted value of spatial sample i is determined by removing sample i from the data vector, solving for the spherical harmonic coefficients, then evaluating the spherical harmonic expansion at the location of sample i . This method can be described mathematically as follows

$$\hat{v}_i(n) = \sum_{p=0}^N \sum_{q=-p}^p a_p^q Y_p^q(P_i) \quad (6.2.11)$$

$$= \vec{y}_{*i}^T(P_i) [\vec{Y}_{*i}^T \vec{Y}_{*i}]^{-1} \vec{Y}_{*i}^T \vec{v}_{*i}(n)$$

where P_i is the position of sample i , and where $\vec{Y}_{*i}$, $\vec{y}_{*i}(P_i)$ and $\vec{v}_{*i}(n)$ are formed by deleting row i from the original vectors or matrices. Note that the predicted value of sample i is determined by a linear combination of the remaining samples and that the weights are data independent.

6.3 Multiple Linear Regression Model

It is desired to obtain the optimal linear predictive model for the EEG data in order to access the adequacy of the bilinear, spherical spline and the truncated spherical harmonic expansion models in describing the spatial relationships in the EEG. This optimal linear model can be determined using multiple linear regression (MLR) approach. The purpose of this section is to review the MLR model along with introducing a method of eliminating the dependence of the model's solution on the activity of the reference.

Due to the large literature base available on MLR (see [6] [7] for an overview), only the MLR model and its solution will be briefly reviewed next. The MLR model assumes that one dependent variable can be predicted by a linear combination of independent variables and a constant term. Using this approach, the activity of spatial sample i is modeled by a linear combination of the activities of the remaining spatial samples as follows

$$\hat{v}_i(n) = \beta_0 + \sum_{\substack{j=1 \\ (j \neq i)}}^{N_i} \beta_{ij} v_j(n) + \epsilon_i(n) \quad (6.3.1)$$

where β_{ij} are the regression coefficients, $\beta_0 = 0$ and $\epsilon_i(n)$ is random variable with the following statistical properties.

$$E\{\epsilon_i(n)\} = 0 \quad \text{Var}\{\epsilon_i(n)\} = \sigma_i^2 \quad \text{Cov}\{\epsilon_i(m), \epsilon_i(n)\} = 0 \quad (m \neq n)$$

It should be noted the MLR model cannot be used for topographic mapping since it can only predict the activity of the EEG at the sampling locations.

For the EEG, it is reasonable to expect that the constant term β_0 in (6.3.1) should equal zero since the temporal average was removed from each channel during preprocessing. In addition, it is not physiologically realistic for any channel to have a non-zero DC value in the recorded data.

With the assumption that $\beta_0 = 0$, it is convenient to rewrite (6.3.1) in matrix form as follows

$$\vec{v}_i = \vec{V}_{*i} \vec{\beta}_i + \vec{\epsilon} \quad (6.3.2)$$

where

$$\vec{v}_i = [\hat{v}_i(1), \hat{v}_i(2), \dots, \hat{v}_i(N_i)]^T,$$

$$\vec{V}_{*i} = [\vec{v}_1, \vec{v}_2, \dots, \vec{v}_{i-1}, \vec{v}_{i+1}, \dots, \vec{v}_{N_i}],$$

$$\vec{\beta}_i = [\beta_{i,1}, \dots, \beta_{i,i-1}, \beta_{i,i+1}, \dots, \beta_{i,N_i}]^T,$$

$$\vec{\epsilon}_i = [\epsilon_i(1), \epsilon_i(2), \dots, \epsilon_i(N_i)]^T,$$

N_i is the number of temporal samples. The least squares solution for $\vec{\beta}_i$, obtained by minimizing the variance of the residual signal with respect to the estimates of $\vec{\beta}_i$, is given by

$$\vec{\beta}_i = (V_{*i}^T V_{*i})^{-1} V_{*i}^T \vec{v}_{*i} \quad (6.3.3)$$

Repeating the regression analysis for all the channels results in N_s sets of regressor coefficients. The elements in the similarity matrix can be written as

$$\vec{S} = \begin{bmatrix} 0 & \beta_{12} & \beta_{13} & \dots & \beta_{1N_s} \\ \beta_{21} & 0 & \beta_{23} & \dots & \beta_{2N_s} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_{N_s,1} & \beta_{N_s,2} & \beta_{N_s,3} & \dots & 0 \end{bmatrix} \quad (6.3.4)$$

Note that the main diagonal elements in $\vec{S}$ are zero due to the predictive nature of the model.

It should be noted that the regression model does not actually meet the criterion for cross-validation since it requires complete knowledge of the activity of the predicted channel. If the number of degrees of freedom in the model and the data set are similar, then the model will tend to overfit the data. Overfitting can be minimized by ensuring that, for each spatial sample, the number of observations N_i in the data set greatly exceeds the number of regressor parameters ($N_i - 1$). Since each epoch consists of 960 sets of spatial measurements ($N_i = 960$) and 31 channels $N_s = 31$, overfitting should be avoided.

6.3.1 Constrained MLR Model

It can be shown that the estimates for the MLR coefficients are affected by activity of the reference, which is an undesirable result. Using a different reference for the signal will generally produce different values for the model's coefficients. Since the MLR model minimizes the variance of the residual signal, the reference sensitivity of the coefficients can be eliminated by making the residual signal independent of the reference. As was discussed in chapter 4, the residual signal is independent of the reference when the elements in each row of the similarity matrix sum to one. This infers that the regressor coefficients of the MLR model should satisfy the following constraint

$$\sum_{j=1}^{N_s} \beta_{ij} = 1 \quad \text{for } 1 \leq i \leq N_s \quad (6.3.5)$$

Solving for the regressor coefficients subject to the above constraint results in the coefficients, along with the residual signal, becoming independent of the effects of the active reference.

The least-squares estimate of the regression coefficients, subject to the constraint $\vec{r}^T \vec{\beta}_c = s$, is given by

$$\vec{\beta}_c = \vec{\beta}_i + (\vec{V}_{*i}^T \vec{V}_{*i})^{-1} \vec{r} \left[\vec{r}^T (\vec{V}_{*i}^T \vec{V}_{*i})^{-1} \vec{r} \right]^{-1} (s - \vec{r}^T \vec{\beta}_i) \quad (6.3.6)$$

where $s = 1$, $\vec{r} = [1, 1, 1, \dots, 1]^T$ (size N_s), and $\vec{\beta}_i$ is defined by (6.3.3) [8]. Equation (6.3.6) determines the linear model with the lowest residual variance subject to the given constraint.

Another modification of the MLR model is to determine reduced rank solutions for the regressor coefficients. By calculating the residual variance as a function of the model's rank, the spatial dimensionality of the EEG can be studied. Reduced rank solutions to the constrained MLR model can be obtained by representing the data matrix $\bar{V}_{*i}$ by its singular value decomposition $\bar{A}\bar{\Sigma}\bar{C}^T$. The rank k solution for the unconstrained MLR model is then given by

$$\bar{\beta}_i = \bar{C}\bar{\Sigma}_k^{-1}\bar{A}^T\bar{v}_{*i} \quad (6.3.7)$$

The rank k solution for the constrained MLR model can be obtained from (6.3.6) and (6.3.7) along with the following result for the matrix inversion shown in the inner brackets in (6.3.6).

$$(\bar{V}_{*i}^T\bar{V}_{*i})^{-1} = \bar{C}^T\bar{\Sigma}_k^{-2}\bar{C} \quad (6.3.8)$$

The MLR model not only provides a lower bound on the residual variance, but can also be used for artifact detection. In general, the MLR model can be considered as a spatial, data-dependent low-pass filter. If the activity of one channel is similar to the activity of the remaining channels, which represents low spatial wavenumbers, then that channel should be predictable and the MLR model should produce a small residual signal. In contrast, localized activity centered about one channel, which contains high spatial wavenumbers, will be generally unpredictable and result in a large residual signal. Reduced rank solutions for the MLR model can be used to vary the sharpness of the spatial filter. However, this application is beyond the purpose of this thesis.

6.4 Quality of Fit

In order to compare the merits of the various models described in this chapter, three statistical parameters were calculated. Two parameters indicate the relative magnitude of the residual variance while the other parameter measures the degree of correlation between the predicted and residual signals. Both the quality of the global and individual channel fits are considered.

A measure of the quality of the global fit for each epoch is given by the normalized residual variance E^2 which is defined as follows.

$$E^2 = \frac{\sum_{n=1}^{N_i} \sum_{k=1}^{N_i} (v_k(n) - \hat{v}_k(n))^2}{\sum_{n=1}^{N_i} \sum_{k=1}^{N_i} v_k^2(n)} \quad (6.4.1)$$

The parameter E^2 represents the proportion of the signal's variance that cannot be accounted for by the model. Small values for E^2 indicates that the model provides a good estimate of the actual signal. Similarly, the normalized residual variance E_i^2 for channel i is defined by

$$E_i^2 = \frac{\sum_{n=1}^{N_i} (v_i(n) - \hat{v}_i(n))^2}{\sum_{n=1}^{N_i} \sum_{k=1}^{N_i} v_k^2(n)} \quad (6.4.2)$$

The parameter E_i^2 indicates the distribution of the residual variance between the channels, and is related to E^2 by

$$E^2 = \sum_{i=1}^{N_i} E_i^2 \quad (6.4.3)$$

A good model for the signal should not only have a small value for E^2 , but also a small correlation between the predicted and residual signals. Models based on a global least-squares solution produce a residual and an estimated signal that are uncorrelated. However, when the model is used for prediction, the residual signal is not guaranteed to be uncorrelated with the predicted signal. In addition, the constrained MLR model can have a non-zero correlation between the predicted and residual signals. One measure of the correlation between the predicted and residual signals is the normalized residual correlation NRC, as defined by

$$NRC = \frac{\sum_{n=1}^{N_i} \sum_{k=1}^{N_i} \hat{v}_k(n) (v_k(n) - \hat{v}_k(n))}{\sum_{n=1}^{N_i} \sum_{k=1}^{N_i} v_k^2(n)} \quad (6.4.4)$$

A zero value for NRC indicates that the residual and predicted signals are uncorrelated while a positive value generally indicates that the predicted signal is too small and a negative value indicates that the predicted signal is too large.

Another parameter that is often used to estimate the quality of fit is the coefficient of determination, as defined by

$$R^2 = \frac{\sum_{n=1}^{N_i} \sum_{k=1}^{N_i} \hat{v}_k^2(n)}{\sum_{n=1}^{N_i} \sum_{k=1}^{N_i} v_k^2(n)} \quad (6.4.5)$$

The parameter R^2 describes the proportion of the signal's variance accounted by the model. It can be shown that R^2 is related to E^2 and NRC by $R^2 = 1 - E^2 - 2*NRC$. Another common statistical parameter is R , which is called the multiple correlation coefficient. In order to prevent presenting redundant results, only the E^2 and E_i^2 parameters will be calculated.

6.5 Determination of the Electrodes' Spherical Coordinates

The spherical spline and truncated spherical harmonic expansion models require knowledge of the electrodes' spherical coordinates. Since the outer surface of the human head is not spherical, an approximation method must be used to convert the electrodes' actual locations into their approximate spherical coordinates.

Estimating the electrodes' spherical coordinates involved first measuring the electrodes xyz-coordinates, then finding the best spherical approximation. Since it is difficult to directly measure the electrodes' positions on the head due to movements by the subject, a fiberglass cap was fabricated for each subject. The positions of the electrodes on the cap where digitized and the best fit sphere determined. Specifically, the following steps were used to determine the electrodes' spherical coordinates.

1. A mold of the each subject's head was constructed using precoated fast setting fiberglass tape (Scotchcast Plus Casting Tape, manufactured by 3M). The mold was then removed from the head and allowed to fully set.
2. The fiberglass mold was placed back on the subject's head, the electrode cap placed over the mold and the electrodes' positions marked.
3. The xyz coordinates of the marked electrode locations were measured with the aid of a numerical milling machine.
4. The center (x_0, y_0, z_0) and radius r of the best fitting spherical shell was determined by minimizing the following expression

$$\sum_{i=1}^{31} |r^2 - (x_i - x_0)^2 - (y_i - y_0)^2 - (z_i - z_0)^2|^2 \quad (6.5.1)$$

with respect to $\{r, x_0, y_0, z_0\}$ where (x_i, y_i, z_i) represents the measured xyz coordinate of the i -th electrode.

5. The electrodes' spherical coordinates were determined using the information obtained from step 4. The coordinate system was then rotated to place z -axis through electrode 17 ($\phi = 0$) while the y -axis was placed in the direction of electrode 30 ($\theta = \pi/2$). This rotation placed all of the electrode position measurements into a common coordinate system.

6. The spherical coordinates for left-right electrode pairs were averaged together in order to account for the left-right symmetry of the head. Note that the electrodes are positioned such that they form pairs on each side of the head, such as electrodes 1 and 2 (refer to Figure 2.4). Let i and j represent, respectively, the electrode pairs on the right and left hand sides of the y -axis. The averaged spherical coordinates are then given by

$$\bar{\phi}_i = \bar{\phi}_j = \frac{\phi_i + \phi_j}{2} \quad (6.5.2)$$

$$\bar{\theta}_i = \frac{\pi}{2} - \delta \quad \bar{\theta}_j = \frac{\pi}{2} + \delta \quad \delta = \frac{\theta_j - \theta_i}{2}$$

The above procedure was used to determine the electrodes' spherical coordinates for five male subjects. Graphical representations of the electrodes' positions are shown in Figures 6.1 and 6.2. Figure 6.1 shows a solid frame representation of the electrodes' actual positions along with the best fitting sphere. As it can be noted from these plots, each subject's

head appears slightly elongated in the frontal area and compressed in the temporal areas. Figure 6.2 shows a radial planar projection of the electrodes' spherical coordinates for the five subjects. As it can be observed, there is consistency in the electrodes' spherical coordinates between the subjects. In addition, both sets of plots indicate that the electrodes sample approximately the surface of a hemisphere.

An averaged set of spherical coordinates were obtained from the five data sets, with the results shown in Table 6.1 and plotted in Figure 6.3. It is these averaged spherical coordinates that were used in the spherical spline and truncated spherical harmonic expansion models.

Sources of error in the above procedure for estimating the spherical coordinates include:

1. Increased head size due to the thickness of the fiberglass mold causing a radial error in the estimated positions of the electrodes. Assuming that the scalp region of interest is approximately spherical, this will introduce insignificant errors.
2. Digitizing errors. The xyz coordinates of the electrodes were determined with a numerically controlled milling machine with an accuracy of ± 0.0002 cm., which far exceeded the accuracy of all other procedures. A much larger error is introduced by the positioning of the pointer used in the digitizing and is estimated to be approximately ± 0.2 cm. .
3. Repeatability of electrode cap placement. Variations in the electrode positions of approximately 1 cm. occur each time cap is placed on the same individual. This introduces an error since the EEG was recorded prior to the head measurement.

A better approach would be to first construct the fiberglass cap, then place the electrodes on the cap and record the EEG, as described in [9]. However this approach would not have allowed the large prerecorded EEG data base to be utilized in this thesis. Fortunately, it was found that slight variations in the estimated positions of the electrodes did not significantly change the results that follow in this thesis.

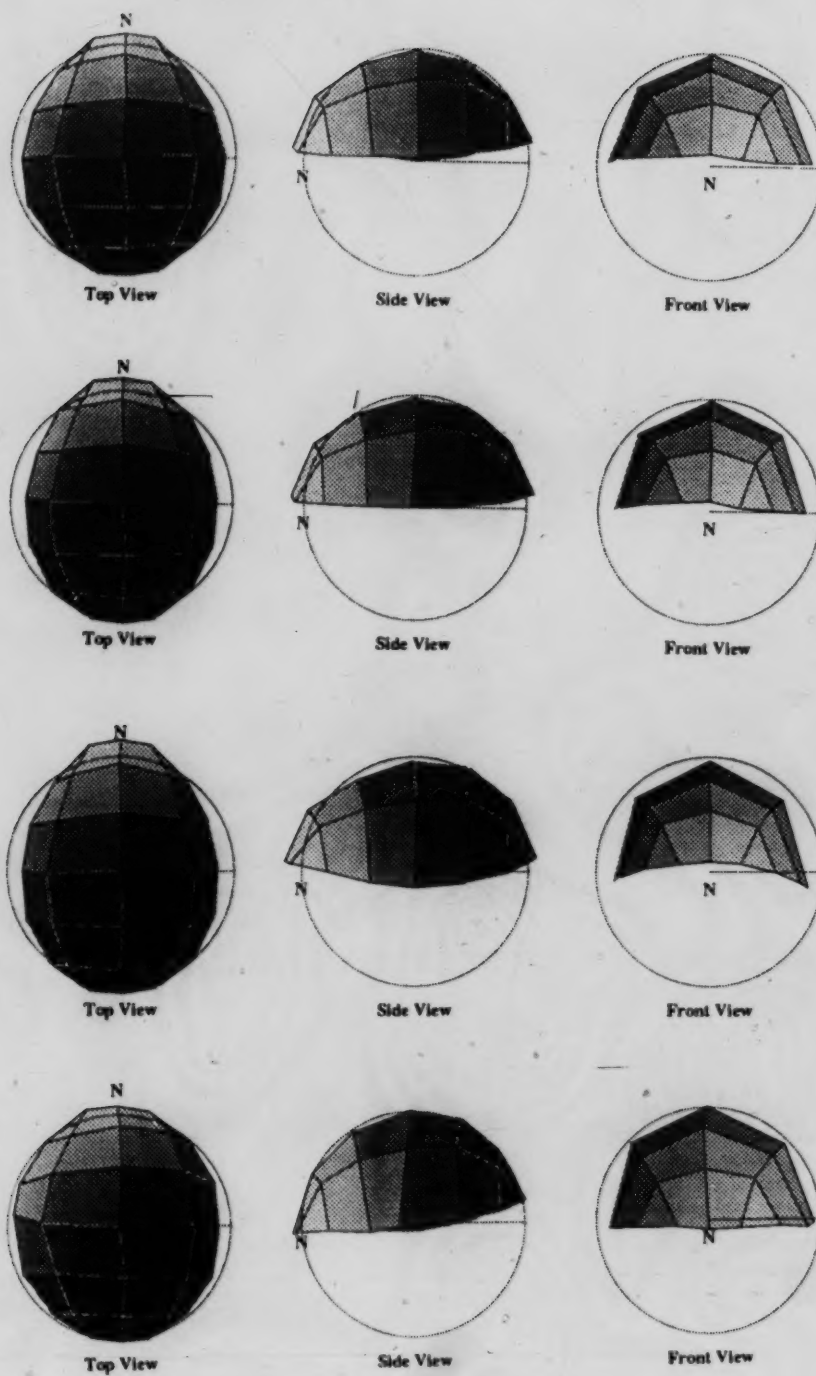


Figure 6.1 - continued on next page

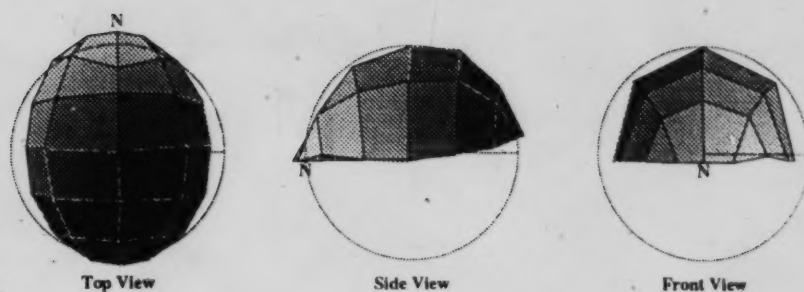


Figure 6.1 - 3-D solid frame representation of the electrodes' actual positions measured from five subjects. The outer boundary of the best fitting sphere is illustrated by the circle while the symbol 'N' indicates the approximate position of the nasion.

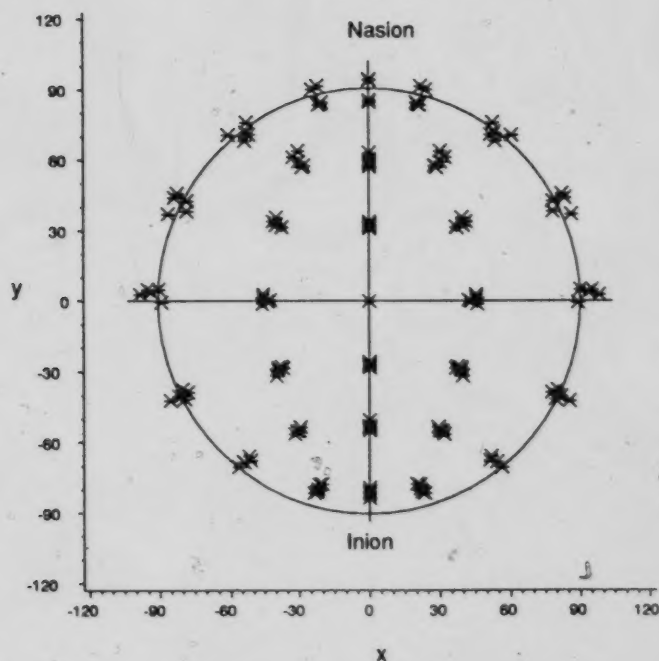


Figure 6.2 Planar projection of the electrodes' estimated spherical coordinates for the five subjects. The planar projection is defined by $x = \phi \cos \theta$ and $y = \phi \sin \theta$ where ϕ is measured in degrees. The circle indicates the boundary of a hemisphere defined by $\phi = \pi/2$.

Nasion

	89.6, 104.1	88.9, 90.0	89.6, 75.9	
90.1, 127.2	67.0, 116.6	59.7, 90.0	67.0, 63.4	90.1, 52.8
91.9, 152.9	51.3, 139.7	32.5, 90.0	51.3, 40.3	91.9, 27.1
93.2, 178.0	44.5, 178.9	0.0, 0.0	44.5, 1.1	93.2, 2.0
90.2, -153.7	48.2, -143.0	27.0, -90.0	48.2, -37.0	90.2, -26.3
86.5, -128.3	62.4, -119.0	53.2, -90.0	62.4, -61.0	86.5, -51.7
	82.4, -105.4	81.8, -90.0	82.4, -74.6	

Inion

Table 6.1 - Averaged spherical coordinates (ϕ, θ), measured in degrees, of the electrodes based on five subjects. The xyz-coordinate system is orientated with the +y axis pointing towards the nasion and the +z axis pointing towards the top of the head.

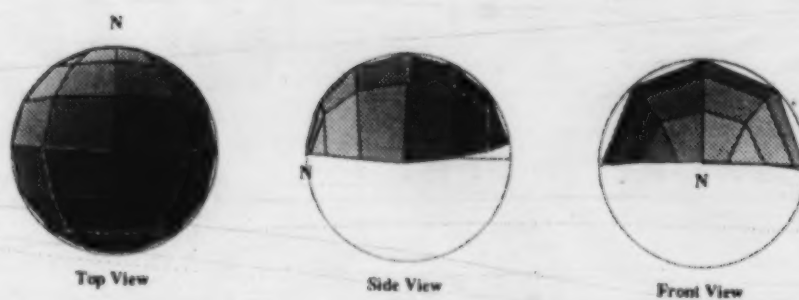


Figure 6.3 - 3-D representation of the averaged electrodes' spherical coordinates shown in Table 6.1

6.6 Experimental Results

This section compares the effectiveness of the bilinear, spherical spline and spherical harmonic expansion models using the method of cross validation applied to EEG data. The parameters E^2 , E_i^2 and NRC were calculated for each model and compared to the optimal results obtained by the constrained MLR model. Finally, several model specific parameters were estimated for the spherical spline and spherical harmonic expansion models.

The experimental data consisted of the 45-epoch EC and EP data sets, as explained in chapter 2. All results presented in this chapter are based on perimeter referenced data.

6.6.1 Constrained MLR Model

For a given data set, the constrained MLR model produces the lowest E^2 obtainable from a reference independent model. Since the constrained MLR model is data dependent, one must specify the data set used to form the model. The coefficients for the constrained MLR model were calculated and the parameters E^2 and NRC determined for the following data groupings with the results summarized in Table 6.2.

1. Individual epochs. The model was determined from and tested on each individual epoch.
2. Combined EC and combined EP data sets. The 45 epochs from the EC data set were combined into one N , by $45N$, sized matrix and the model determined, then tested on both the EC and EP data sets. The same procedure was repeated for the model formed from the EP data set.
3. Combined EC and EP data set. The model was determined from a N , by $30N$, sized matrix formed by selecting the first epoch from each subject's EC and EP recordings. Only one epoch from each individual's EEG was selected to reduce the computational demands. The single model was then tested on both the EC and EP data sets.

From the results shown in Table 6.2, it is evident that E^2 is larger for the EP data set than the EC data set. This result is consistent with the results obtained from the circular array data in chapter 5. That is, the spatial bandwidth of the EP data is larger than the EC data, thus increasing the amount of the signal that has been spatially aliased. An aliased signal component can be considered as being unpredictable assuming that the model is not overfitting the data.

It can also be observed from the results in Table 6.2 that the lowest E^2 values are obtained for the MLR model determined from the individual epochs. Basing the model on combined epoch data matrices results in E^2 increasing by approximately 50%. The similarity of the results in the last three rows for each data set infers that the spatially predictable part of both data sets is also similar.

The next set of results shown in Figure 6.4 illustrates the effect on E^2 due to varying the rank of the constrained MLR model. In order to reduce the computational load only one epoch from each subject's EEG was selected, resulting in two 15-epoch data sets. Various rank k constrained MLR models were formed and tested on the same 15-epoch data sets.

	EC Data Set		EP Data Set	
	$E^2 \times 100$	NRC $\times 100$	$E^2 \times 100$	NRC $\times 100$
Individual	3.3 ± 1.4	-0.2 ± 0.4	6.7 ± 2.5	-0.2 ± 0.2
Combined EC	5.0 ± 1.7	-0.3 ± 1.2	10.6 ± 4.5	-2.9 ± 2.4
Combined EP	5.7 ± 1.5	0.8 ± 0.6	9.2 ± 3.2	-0.4 ± 1.2
Combined EC and EP	5.2 ± 1.6	0.4 ± 0.8	9.4 ± 3.5	-1.2 ± 1.3

Table 6.2 - Mean and standard deviation of E^2 and NRC for the constrained full-rank MLR model based on individual and various combined data sets. The small, but non-zero values for the NRC are due to the use of a constrained model.

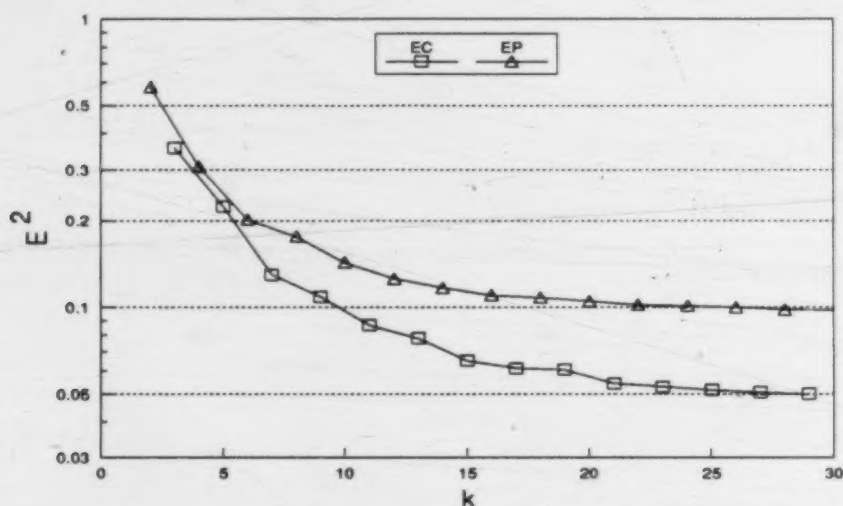


Figure 6.4 - E^2 vs. the rank k for the constrained MLR model determined from and tested on the same 15-epoch data set.

From the results shown in Figure 6.4, it can be noted that E^2 decreases as the rank increases. Due to the use of least squares and due to the fact that the model was formed and tested on the same data sets, E^2 will decrease as the rank is increased. As the rank increases beyond 15 and 20 for the EP and EC data sets respectively, E^2 approaches an approximately constant value and provides an estimate of the spatial dimensionality of the EEG.

It is rather interesting that the estimated spatial dimensionality of the predictable part of the EC data appears slightly higher than the EP data, although the spatial bandwidth of the EC data set was lower than spatial bandwidth of the EP data set. One possible explanation for this result is to remember that E^2 represents the unpredictable component of the signal and that E^2 is larger for the EP data set than for the EC data set. As the rank of the model increases, E^2 reaches a lower bound quicker for the EP data set due to its larger final value, as compared to the EC data set.

A better test of the spatial dimensionality would be to determine and test the model on different data sets. Unfortunately sufficient data was not available to take this approach. As a compromise, the constrained MLR model was formed and tested using different epochs from each individual. In addition, the MLR model obtained from the EC data set was also tested on the EP data set and vice versa. It was found that all of the E^2 vs. rank plots were similar to the plots shown in Figure 6.4.

6.6.2 Spherical Spline Model

The spherical spline model requires specification of the parameters λ and κ . Shown in Table 6.3 are the values of the parameters E^2 and NRC obtained for spherical spline model with κ ranging from 2-5 for $\lambda = 0$. From these results, it can be noted that the optimal value for κ is 2.

Positive values for λ were also considered (results not shown) for κ ranging from 2-5. For each value of κ , it was found that the optimal value for λ that produced the lowest E^2 value produced results similar to that obtained for $\lambda = 0$ and $\kappa = 2$. However, for applications such as topographic mapping, it is desirable to have $\lambda = 0$, since this results in an interpolating function.

The use of the spherical spline model for the spatial EEG, along with the determination of κ , has also been considered by Perrin et al. [10]. These authors estimated the optimal value of κ to be 4 (λ was assumed to be zero), based on the results obtained from a set of randomly located dipole generators located inside a physical model of the head. Since the optimal value for κ found in this section is based on actual data, it is concluded that κ should be set to 2.

κ	EC Data Set		EP Data Set	
	$E^2 \times 100$	NRC $\times 100$	$E^2 \times 100$	NRC $\times 100$
2	7.4 ± 1.5	1.1 ± 0.6	10.6 ± 3.5	-0.2 ± 1.4
3	9.3 ± 1.7	0.6 ± 0.4	11.5 ± 3.1	0.0 ± 0.9
4	14.6 ± 2.9	1.4 ± 0.3	16.2 ± 3.0	1.1 ± 0.6
5	21.9 ± 4.1	0.5 ± 0.3	23.0 ± 3.2	0.2 ± 0.4

Table 6.3 - Mean and standard deviation of E^2 and NRC vs. κ for the spherical spline model with $\lambda = 0$.

6.6.3 Spherical Harmonic Expansion Model

Shown in Table 6.4 are the statistical results obtained for the spherical harmonic expansion model. In order to ensure that the model was overdetermined, only expansion degrees (N) ranging from 0 to 4 were considered. Although E^2 was similar for the $N = 2$ and $N = 3$ models, the 2nd degree model had a smaller NRC along with having fewer coefficients (9 vs. 16). It is therefore concluded that the optimal degree of expansion for the spherical harmonic model is $N = 2$.

N	EC Data Set		EP Data Set	
	$E^2 \times 100$	NRC $\times 100$	$E^2 \times 100$	NRC $\times 100$
0	97.0 $\pm$ 2.6	-3.1 $\pm$ 0.1	94.8 $\pm$ 1.8	-3.1 $\pm$ 0.1
1	28.2 $\pm$ 5.2	-3.8 $\pm$ 6.9	28.8 $\pm$ 3.6	-3.9 $\pm$ 0.5
2	14.6 $\pm$ 3.3	-4.5 $\pm$ 1.1	16.1 $\pm$ 3.3	-4.9 $\pm$ 1.0
3	14.1 $\pm$ 3.9	-8.0 $\pm$ 3.3	16.4 $\pm$ 4.4	-9.4 $\pm$ 2.6
4	61.6 $\pm$ 18.3	-53 $\pm$ 17	73 $\pm$ 38	-67 $\pm$ 37

Table 6.4 - Mean and standard deviation of E^2 and NRC for the spherical harmonic expansion model of degree N . The large values of E^2 for $N=0$ is due to the use of the perimeter reference, which removes most of the spatially constant value from the data.

6.6.4 Bilinear Model

The statistical results for the bilinear model are shown in Table 6.5. Note that the bilinear model has no data dependent parameters. The most relevant feature of the results is the relatively large positive values for NRC, which infers that the model tends to underestimate the actual magnitude of the signal.

EC Data Set		EP Data Set	
$E^2 \times 100$	NRC $\times 100$	$E^2 \times 100$	NRC $\times 100$
11.4 $\pm$ 1.4	9.5 $\pm$ 1.6	14.4 $\pm$ 3.3	9.0 $\pm$ 1.6

Table 6.5 - Mean and standard deviation of E^2 and NRC for the bilinear model. Note the relatively large positive value for E^2 , which indicates that the model is underestimating the magnitude of the data.

6.6.5 Comparison and discussion of results

A comparison of the optimal results obtained from the constrained MLR, spherical spline, bilinear and truncated spherical harmonic expansion models are shown in Table 6.6. The results from the constrained MLR model based on the combined epochs represent the lower bound for E^2 obtainable from a predicting linear model. Inspecting the last three rows of results, the spherical spline model clearly produces the lowest E^2 and NRC values, and provides a near optimal result for the EP data set.

In order to produce a topographic map of the spatial EEG activity, one could use either the spherical spline, bilinear or the truncated spherical harmonic models to estimate the activity of the EEG between channels. It is desirable to estimate the error in these topographic maps. The E^2 values shown in Table 6.6 are for cross-validated models that predict the activity of each channel as a linear combination of the activities of the remaining channels. Let the spatial prediction distance of the model be defined as the distance from the prediction point to the nearest channel. It is reasonable to expect that the error in the prediction will increase as the prediction distance increases. The average spatial prediction distance for the cross-validated model is the average interelectrode distance. When the activity of all of the channels are used to produce a topographic map, the largest prediction distance is approximately $1/\sqrt{2}$ times the average interelectrode separation. Thus, one can conclude that the largest errors in topographic maps of the EEGs activity will be less than the E^2 values shown in Table 6.6.

The next set of results (Tables 6.7-6.14 and Figures 6.5-6.12) shows the mean and standard deviation of E_i^2 obtained from the various models. Recall that E_i^2 indicates the distribution of the residual variance amongst the channels. For the results (Tables 6.7,9,11,13) derived from the EC data set, the largest residual variances occur in the occipital region of the head. It is speculated that the increased variance in the occipital region is due to alpha activity since it will be shown that the EC data set had significant alpha activity and since alpha activity tends to be strongest in the occipital region.

From the results (Tables 6.8,10,12,14) derived from the EP data set, one can observe a more even distribution of the variance, with the largest variances occurring at those electrodes located nearest the ears. Upon close inspection of the EEG tracings for the EP data set, the electrodes nearest the ears tended to have more muscle artifacts. Muscle artifacts are spatially localized and will thus will increase the associated values of E_i^2 .

Shown in Figure 6.14 are the composite temporal PSDF estimates of the EC and EP data sets. The major difference between the two spectrums is the peak around 10 Hz. in the EC spectrum due to alpha activity. Again, inspection of the EC epochs confirmed the presence of alpha activity. This alpha activity was suppressed during the EP recording.

The normalized residual spectral estimate for the constrained MLR, spherical spline, bilinear and spherical harmonic expansion models are shown in Figure 6.13. (refer to chapter 5 for calculation details). The constrained MLR model was formed and tested on the same 45-epoch data sets. Each model produced a residual variance that reaches a minimum in the frequency range of 8-11 Hz and 6-8 Hz for the EC and EP data sets respectively. The increasing residual variance as the frequency approaches 0 is again believed to be due to the phase differences of the analog filters. As the frequency increases beyond approximately 10 Hz, the increasing residual variance is believed to be due to the increasing spatial bandwidth of the signal, as discussed in the previous chapter. The

increasing spatial bandwidth increases the amount of the signal that is aliased, and thus unpredictable. Also, the peak in the residual spectra at 30 Hz is a sub harmonic of 60 Hz., and is thus believed to be an artifact.

The constrained MLR model produced the smallest residual variance in the spectrums, except for the EC spectrum above 25 Hz. The spherical spline model produced a slightly larger residual variance when compared to the MLR models residual spectrum, with the largest difference occurring in the alpha band of frequencies for the EC data set. It is suggested that the MLR model is dominated by a consistent spatial pattern in the alpha band since a significant amount of the signal's variance is contained in that band. Thus, the MLR model can reduce the residual variance in the alpha band, at the expense of having a slightly larger residual variance at higher frequencies (>25 Hz.).

Model	EC Data Set		EP Data Set	
	$E^2 \times 100$	NRC $\times 100$	$E^2 \times 100$	NRC $\times 100$
MLR (individual)	3.3 ± 1.4	-0.2 ± 0.4	5.0 ± 1.6	-0.2 ± 0.2
MLR (combined)	5.0 ± 1.7	-0.3 ± 1.2	9.2 ± 3.2	-0.4 ± 1.2
Spline ($\kappa = 2, \lambda = 0$)	7.4 ± 1.5	1.1 ± 0.6	10.6 ± 3.5	-0.2 ± 1.4
Bilinear	11.4 ± 1.7	9.5 ± 1.6	14.4 ± 3.3	9.0 ± 1.6
SHA (N=2)	14.6 ± 3.3	-4.5 ± 1.1	16.1 ± 3.3	-4.9 ± 1.0

Table 6.6 - Mean and standard deviation of E^2 and NRC based on the various prediction models. The first row of results are for the constrained MLR model based on the individual epochs while the second row of results are based on the constrained MLR model determined from the combined 30-epoch data set and tested on the combined 45-epoch data set.

	Nasion			
	0.10 ± 0.09	0.08 ± 0.08	0.11 ± 0.11	
0.16 ± 0.12	0.07 ± 0.05	0.08 ± 0.10	0.08 ± 0.07	0.20 ± 0.21
0.14 ± 0.10	0.12 ± 0.09	0.16 ± 0.10	0.11 ± 0.06	0.17 ± 0.12
0.14 ± 0.11	0.13 ± 0.07	0.11 ± 0.06	0.15 ± 0.08	0.18 ± 0.15
0.18 ± 0.14	0.08 ± 0.03	0.12 ± 0.04	0.10 ± 0.05	0.24 ± 0.20
0.31 ± 0.20	0.16 ± 0.10	0.38 ± 0.33	0.19 ± 0.12	0.43 ± 0.20
	0.18 ± 0.08	0.10 ± 0.05	0.25 ± 0.18	

Table 6.7.- Mean and standard deviation of $E_i^2 \times 100$ for the constrained MLR model formed and tested on the 45-epoch EC data set.

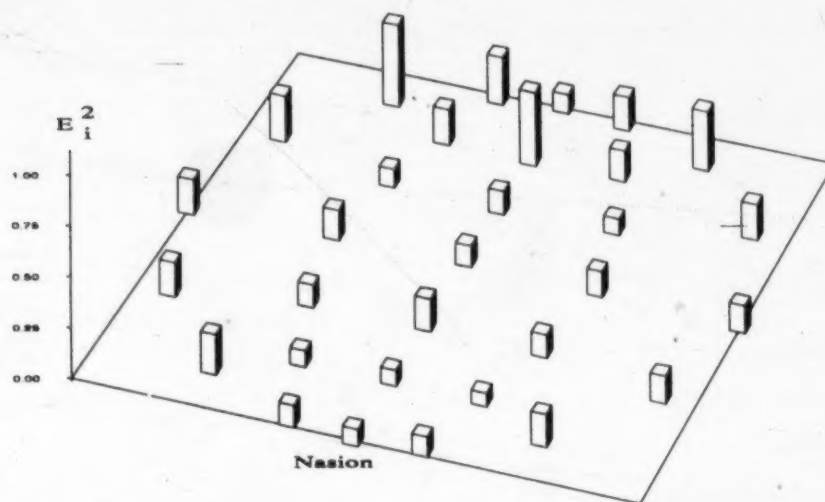


Figure 6.5 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.7.

	0.26 ± 0.15	0.21 ± 0.18	0.30 ± 0.32	
0.56 ± 0.48	0.17 ± 0.09	0.19 ± 0.16	0.25 ± 0.30	0.48 ± 0.33
0.49 ± 0.30	0.24 ± 0.12	0.38 ± 0.19	0.27 ± 0.18	0.46 ± 0.20
0.51 ± 0.59	0.27 ± 0.23	0.23 ± 0.13	0.27 ± 0.12	0.56 ± 1.12
0.42 ± 0.39	0.14 ± 0.05	0.19 ± 0.05	0.14 ± 0.06	0.39 ± 0.19
0.30 ± 0.12	0.16 ± 0.07	0.29 ± 0.17	0.19 ± 0.13	0.37 ± 0.16
	0.17 ± 0.07	0.13 ± 0.04	0.23 ± 0.15	

Table 6.8 - Mean and standard deviation of $E_i^2 \times 100$ for the constrained MLR model formed and tested on the 45-epoch EP data set

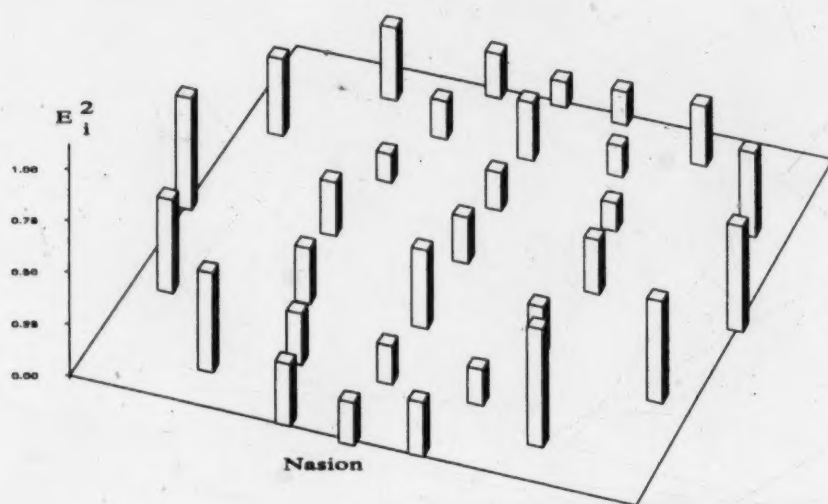


Figure 6.6 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.8.

	0.11 ± 0.10	0.10 ± 0.10	0.13 ± 0.14	
0.18 ± 0.13	0.08 ± 0.06	0.13 ± 0.12	0.10 ± 0.09	0.22 ± 0.23
0.15 ± 0.10	0.15 ± 0.10	0.22 ± 0.18	0.13 ± 0.07	0.19 ± 0.15
0.15 ± 0.11	0.17 ± 0.09	0.13 ± 0.06	0.19 ± 0.11	0.20 ± 0.16
0.31 ± 0.19	0.11 ± 0.04	0.35 ± 0.30	0.13 ± 0.07	0.48 ± 0.23
0.56 ± 0.48	0.23 ± 0.15	0.67 ± 0.88	0.28 ± 0.21	0.78 ± 0.52
	0.23 ± 0.17	0.17 ± 0.08	0.38 ± 0.39	

Table 6.9 - Mean and standard deviation of $E_i^2 \times 100$ for the EC perimeter referenced data set based on the spherical spline model ($\kappa = 2$, $\lambda = 0$).

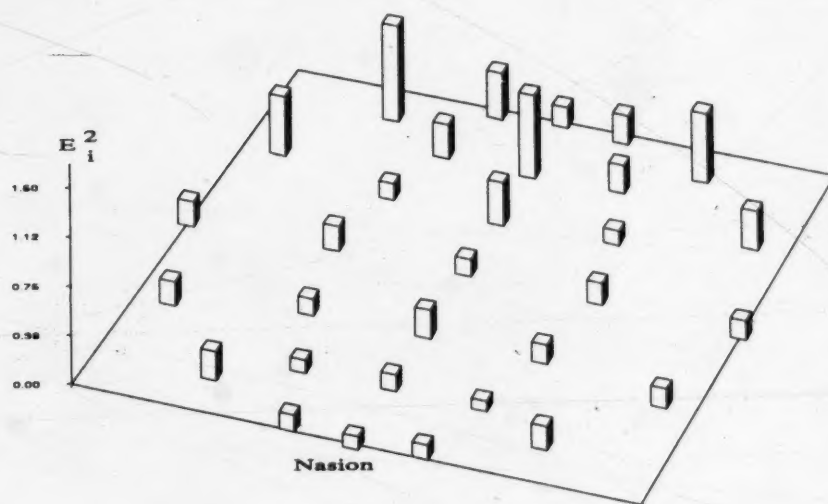


Figure 6.7 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.9.

	0.28±0.16	0.24±0.22	0.33±0.38	
0.60±0.51	0.21±0.11	0.28±0.16	0.29±0.30	0.51±0.34
0.54±0.39	0.29±0.16	0.50±0.17	0.30±0.20	0.50±0.44
0.53±0.58	0.34±0.33	0.26±0.13	0.34±0.21	0.60±1.17
0.45±0.45	0.15±0.05	0.29±0.15	0.15±0.06	0.46±0.34
0.35±0.13	0.18±0.09	0.42±0.33	0.23±0.14	0.45±0.22
	0.18±0.07	0.15±0.06	0.26±0.19	

Table 6.10 - Mean and standard deviation of $E_i^2 \times 100$ for the perimeter referenced EP data set based on the spherical spline model.

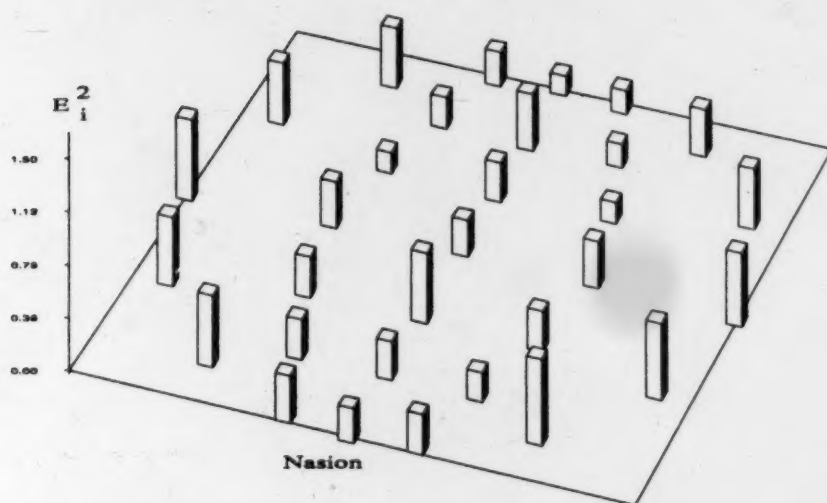


Figure 6.8 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.10.

	0.15±0.11	0.10±0.09	0.17±0.16	
0.23±0.15	0.13±0.09	0.13±0.12	0.17±0.12	0.27±0.23
0.19±0.12	0.28±0.15	0.33±0.26	0.24±0.15	0.25±0.15
0.24±0.12	0.34±0.17	0.30±0.15	0.41±0.20	0.27±0.17
0.30±0.18	0.23±0.09	0.41±0.18	0.28±0.12	0.42±0.22
0.99±0.74	0.39±0.25	1.10±1.28	0.46±0.34	1.36±0.86
	0.44±0.31	0.22±0.12	0.63±0.53	

Table 6.11 - Mean and standard deviation of $E_i^2 \times 100$ for the perimeter referenced EC data set based on the bilinear model.

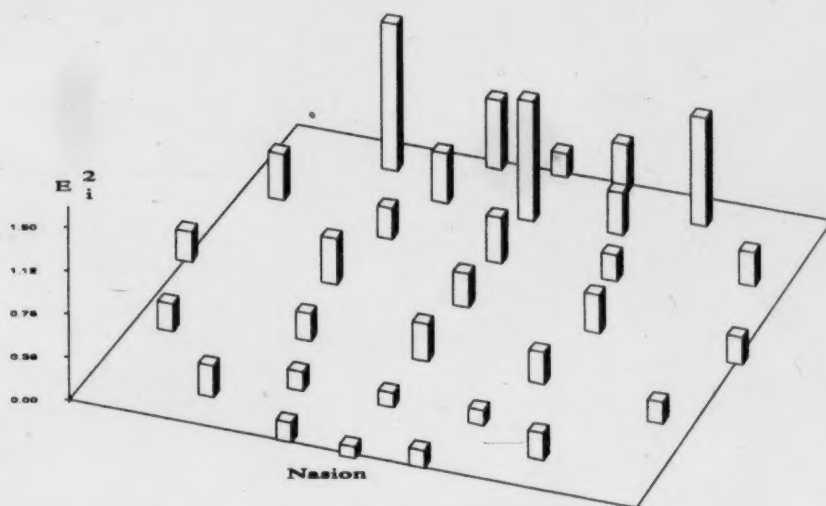


Figure 6.9 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.11.

	0.36±0.17	0.24±0.20	0.39±0.34	
0.70±0.54	0.31±0.15	0.27±0.19	0.41±0.30	0.59±0.34
0.56±0.35	0.54±0.24	0.74±0.26	0.45±0.19	0.53±0.34
0.59±0.55	0.61±0.61	0.49±0.25	0.67±0.41	0.65±1.21
0.49±0.42	0.31±0.10	0.41±0.17	0.33±0.13	0.45±0.25
0.54±0.19	0.27±0.13	0.61±0.48	0.36±0.21	0.70±0.34
	0.27±0.10	0.18±0.09	0.38±0.28	

Table 6.12 - Mean and standard deviation of $E_i^2 \times 100$ for the perimeter referenced EP data set based on the bilinear model.

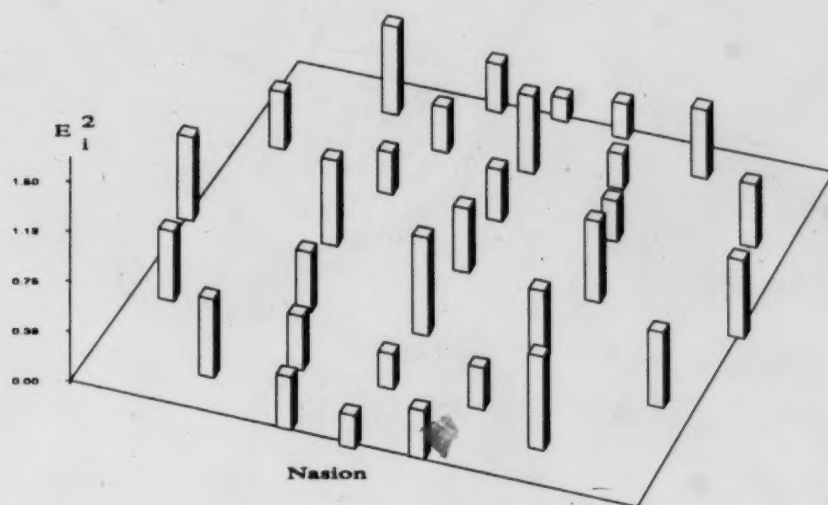


Figure 6.10 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.12.

	0.21±0.10	0.19±0.09	0.23±0.13	
0.36±0.13	0.17±0.06	0.19±0.14	0.19±0.07	0.41±0.24
0.37±0.15	0.23±0.12	0.31±0.13	0.24±0.11	0.41±0.16
0.55±0.20	0.39±0.13	0.19±0.07	0.41±0.16	0.61±0.25
0.59±0.25	0.31±0.12	0.30±0.14	0.35±0.24	0.77±0.32
1.14±0.80	0.42±0.21	1.00±1.16	0.48±0.33	1.39±0.76
	0.67±0.41	0.66±0.27	0.85±0.74	

Table 6.13 - Mean and standard deviation of $E_i^2 \times 100$ for the perimeter referenced EC data set based on the spherical harmonic expansion model ($N=2$).

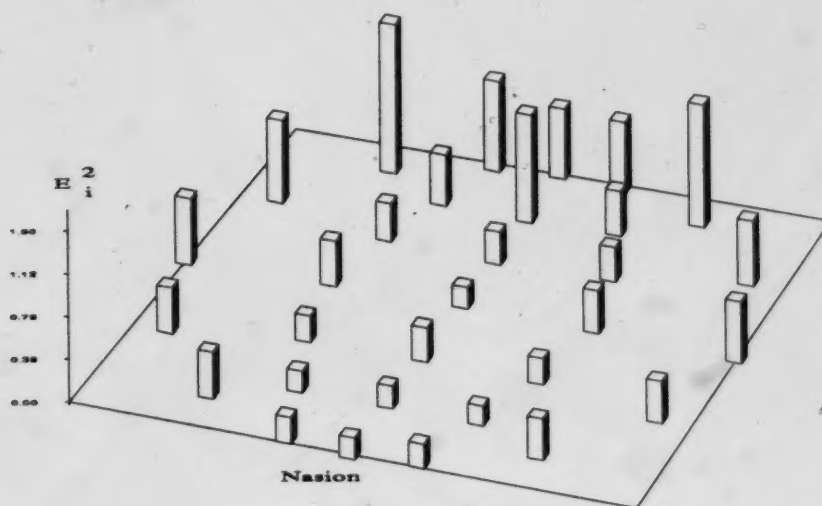


Figure 6.11 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.13.

	0.41 ± 0.15	0.37 ± 0.17	0.44 ± 0.30	
0.82 ± 0.52	0.29 ± 0.12	0.35 ± 0.15	0.36 ± 0.28	0.70 ± 0.37
0.72 ± 0.31	0.39 ± 0.18	0.51 ± 0.16	0.41 ± 0.22	0.68 ± 0.28
0.82 ± 0.54	0.58 ± 0.41	0.35 ± 0.11	0.57 ± 0.27	0.90 ± 1.23
0.69 ± 0.44	0.30 ± 0.09	0.32 ± 0.08	0.30 ± 0.10	0.69 ± 0.19
0.68 ± 0.23	0.34 ± 0.14	0.62 ± 0.43	0.40 ± 0.18	0.79 ± 0.31
	0.40 ± 0.15	0.40 ± 0.12	0.48 ± 0.32	

Table 6.14 - Mean and standard deviation of $E_i^2 \times 100$ for the perimeter referenced EP data set based on the spherical harmonic expansion model ($N=2$).

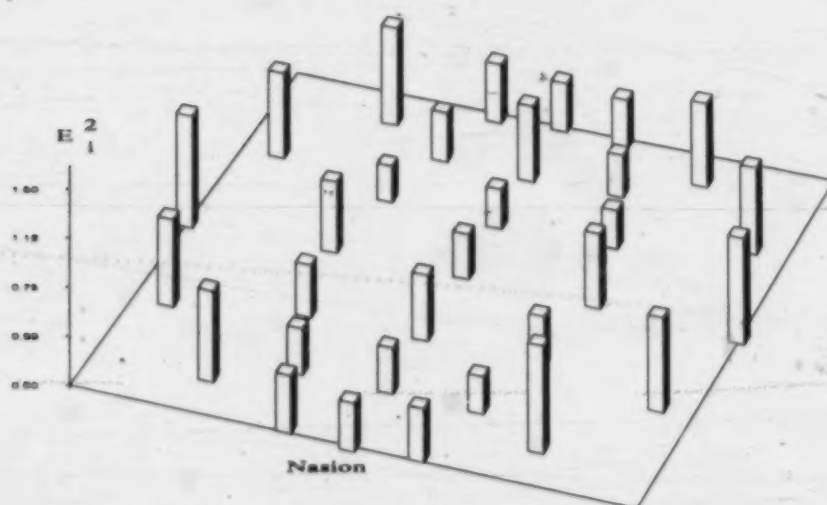
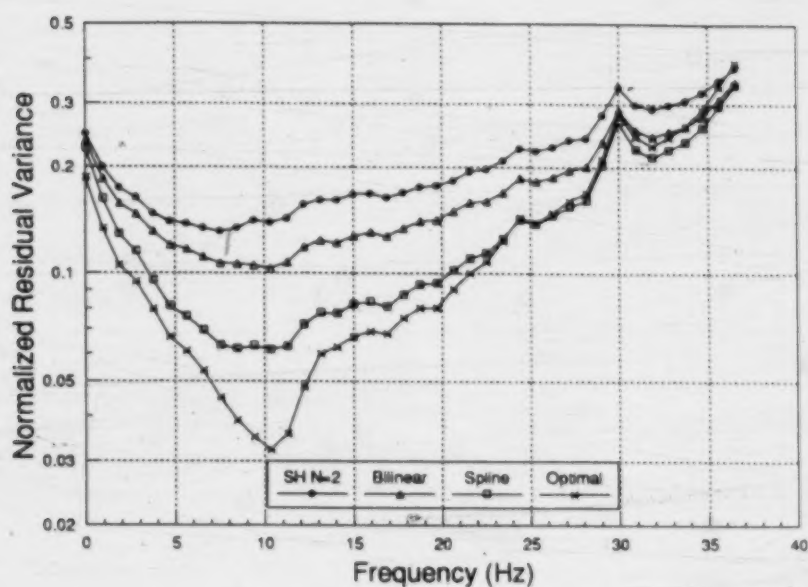
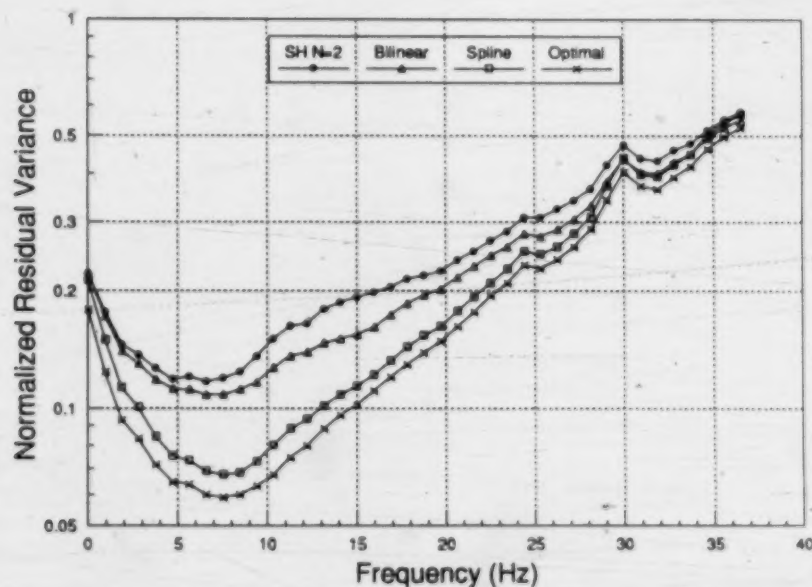


Figure 6.12 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 6.14.



(a) EC data set



(b) EP data set

Figure 6.13 - Normalized residual variance vs. frequency for the EC and EP data set for the constrained MLR (optimal), spherical spline ($\kappa = 2$, $\lambda = 0$), bilinear and spherical harmonic (SH) expansion ($M=2$) models. The MLR model was formed and tested on the same composite 45-epoch data sets. Refer to section 3.2 in chapter 5 for calculation details.

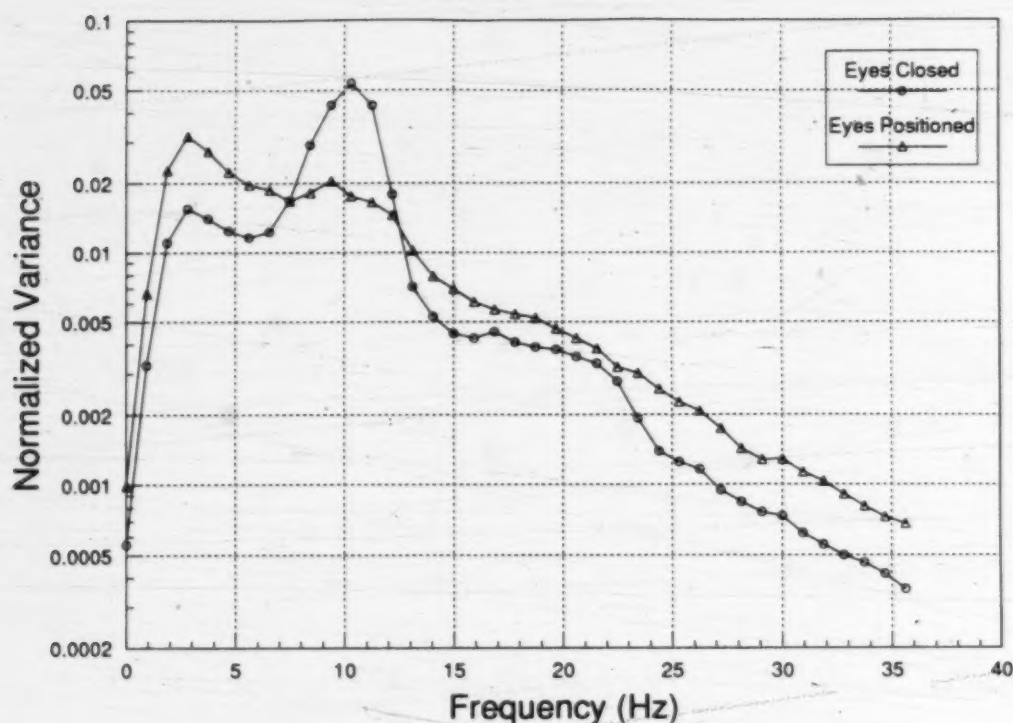


Figure 6.14 - Composite temporal PSDF estimates for the perimeter referenced EC and EP data sets. Refer to section 2 in chapter 5 for calculation details.

6.7 Conclusions

An optimal prediction model for the EEG data sets was developed that is based on a constrained MLR model. The model is constrained to produce a residual signal that is unaffected by the activity of the reference signal and can be considered as a spatial filter. It was also suggested that the sharpness of the spatial filter could be controlled by varying the rank of the constrained MLR model. The results produced by this model was used as a benchmark to compare the adequacy of various other linear models.

Based on the experimental results, the spherical spline model provides a near optimal method of prediction for the spatial EEG. The results also showed that the optimal parameters for the spherical spline model were $\kappa = 2$ and $\lambda = 0$ while the optimal degree of expansion for the spherical harmonic model was $N = 2$.

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7 Spherical Harmonic Analysis of the EEG

An alternative representation of the EEG is to transform the data into a set of time-varying spherical harmonic coefficients. At each sampling time, the spherical harmonic expansion of the EEG is defined by a set of spherical harmonic coefficients. The analysis of the spherical harmonic coefficients sequences, instead of the original EEG, might lead to further insights into the properties of the signal.

The justification of representing the EEG by its spherical harmonic expansion is based on the results from the previous chapter. It was shown that a large percentage of the EEG's variance could be predicted by a low degree spherical harmonic expansion. Since a low degree expansion involves only low spatial frequencies, the gross features of the signal can be investigated by analyzing the spherical harmonic coefficients.

Two reasons for representing the EEG by a spherical harmonic expansion are; to reduce the dimensionality of the data and to extract new information from the data. Dimensionality reduction, achieved by limiting the degree of expansion, simplifies the analysis of the data. Basing the expansion on the sampled spherical harmonic functions has the advantage that the functions have known properties which are related to potential theory. As discussed in chapter 2, the spherical harmonic expansion of a function is equivalent to a multipole expansion about the origin of a sphere. More importantly, since the spherical harmonic functions represent spherical waves, wave activity in the EEG can be detected by investigating the spherical harmonic coefficient sequences. Thus, the spherical harmonic expansion of the data might reveal new features in the data.

Two purposes of this chapter are; to determine the validity of representing the EEG by its spherical harmonic expansion and to determine if any new insights into the EEG can be obtained by analyzing the temporal variations in the spherical harmonic coefficient sequences. The first section in this chapter considers the representation of the EEG by a low degree spherical harmonic expansion, which is equivalent to a reduced rank projection of the data. In order to determine the adequacy of the spherical harmonic model in representing the EEG, the optimal reduced rank projection of each EEG data set was also calculated. The residual variances obtained from the reduced rank projections along with the basis vectors produced by both methods are compared. The next section investigates the variances of the spherical harmonic coefficient sequences calculated from the EEG data sets. In addition, estimates of the coefficient sequences' spatial and temporal PSDF are obtained. The final section investigates the time-evolution of the coefficient sequences for one alpha dominant EEG epoch by plotting the Lissajous trajectories.

7.1 Smoothing of the EEG by a Spherical Harmonic Expansion

The purpose of this section is to test the adequacy of representing the spatial EEG by a spherical harmonic expansion, which is equivalent to a reduced rank projection of the data based on sampled spherical harmonic functions. In order to test the adequacy of the expansion, the optimal reduced rank projection for each data set is also determined.

7.1.1 Reduced Rank Models

Assume that the observed data is to be approximated by a linear combination of basis vectors, as given by

$$\vec{V} = \vec{V}_p = \vec{B}\vec{K} \quad (7.1.1)$$

where

$$\vec{V} = [\vec{v}(1), \vec{v}(2), \dots, \vec{v}(N_t)],$$

$\vec{v}(i)$ is a N_s sized vector consisting of the spatial EEG data at sampling interval i , N_s and N_t are the number of spatial and temporal samples respectively, and where the i -th row in the $N_s \times N_t$ sized matrix $\vec{K}$ represents the weighting values for the i -th basis vector. The basis vectors are represented in matrix form by

$$\vec{B} = [\vec{b}_1, \vec{b}_2, \dots, \vec{b}_P] \quad (7.1.2)$$

where $\vec{b}_i$ is a N_s sized basis vector and where the number of basis vectors P is less than N_s . By minimizing the sum of the squared differences between $\vec{V}$ and $\vec{V}_p$ with respect to the weighting matrix, the solution for $\vec{K}$ is given by

$$\vec{K} = [\vec{B}^T \vec{B}]^{-1} \vec{B}^T \vec{V} \quad (7.1.3)$$

The least-squares rank P projection of the data matrix is then given by

$$\begin{aligned} \vec{V}_p &= \vec{B} [\vec{B}^T \vec{B}]^{-1} \vec{B}^T \vec{V} \\ &= \vec{S} \vec{V} \end{aligned} \quad (7.1.4)$$

where $\vec{S}$ is the similarity matrix discussed in chapter 3. It can be observed from the above equation that the rank P projection of the data matrix only requires specification of P basis vectors.

7.1.2 Optimal Reduced Rank Projection

The Karhunen-Loève expansion (KLE) [1] determines an optimal set of basis vectors for a given data matrix. Optimality is defined as that set of orthonormal basis vectors that accounts for the largest amount of variance in the data. Since these optimal basis vectors are derived directly from the data and are not constrained by any spatial structure, information about the locations of the spatial samples is not required. As a result, each data set will generally have a unique set of optimal basis vectors.

The KLE is directly related to the singular value decomposition (SVD). The SVD can be used to project the data matrix $\bar{V}$ into a lower dimensional space in an optimal least-squares sense. Assuming that $N_i \geq N_r$, the SVD of $\bar{V}$ can be written as

$$\bar{V} = \bar{A} \bar{\Sigma} \bar{C}^T \quad (7.1.5)$$

where $\bar{A}$ and $\bar{C}$ are respectively N_r by N_r and N_i by N_i sized orthonormal matrices and $\bar{\Sigma}$ is a diagonal matrix. The columns of $\bar{A}$ are the eigenvectors of $\bar{V}\bar{V}^T$ while the diagonal elements in $\bar{\Sigma}$ are the square roots of the eigenvalues of $\bar{V}\bar{V}^T$ and are referred to as the singular values. The singular values are sorted into descending order, with the largest value placed in the upper left-hand corner of $\bar{\Sigma}$. The columns of $\bar{C}$ are the eigenvectors associated with the N_i non-zero eigenvalues of $\bar{V}^T\bar{V}$. Due to the reduced size of $\bar{C}$, the above factorization should strictly be called the singular value factorization.

The optimal rank P projection of $\bar{V}$ can be obtained by

$$\bar{V}_P = \bar{A} \bar{\Sigma}_P \bar{C}^T \quad (7.1.6)$$

where $\bar{\Sigma}_P$ is obtained by setting all but the P largest eigenvalues in $\bar{\Sigma}$ to zero. The sum of the squared errors (SSE) between $\bar{V}$ and $\bar{V}_P$ can be shown to be equal to

$$SSE = \sum_{n=P+1}^{N_i} \lambda_n^2 \quad (7.1.7)$$

where λ_n is the n -th singular value in $\bar{\Sigma}$. By using the eigenvectors associated with the largest singular values, the SSE for a rank P projection can be minimized. This rank P projection of $\bar{V}$ is optimal in the sense that it has the smallest SSE value of all rank P projections.

The SVD can be considered as decomposing the data matrix into a weighted set of spatial basis vectors as follows

$$\bar{V}_P = \bar{A} \bar{K} \quad (7.1.8)$$

where $\bar{K} = \bar{\Sigma}_P \bar{C}^T$ (size N_r by N_i). Upon comparing the above equation with (7.1.1), it can be observed that the columns in $\bar{A}$ represent the basis vectors while the rows in $\bar{K}$ represent the associated weights of each basis vector. The eigenvectors contained in $\bar{A}$ are commonly referred to as the principal components (PCs). Equation (7.1.8) is also the KLE of the data matrix. Using the orthonormal property of the eigenvectors, the reduced rank projection of the data matrix can be written as

$$\bar{V}_P = \bar{A}_P \bar{A}_P^T \bar{V} \quad (7.1.9)$$

where $\bar{A}_P$ is formed by using only the first P eigenvectors contained in $\bar{A}$.

Calculation of the eigenvectors via the SVD is numerically time consuming for large sized matrices. In order to increase the speed of the calculations, the eigenvectors can be calculated directly from the covariance matrix $\overline{VV^T}$ [2]. Although calculation of the eigenvectors via either the SVD or the direct method are mathematically identical, the SVD approach is less sensitive to the effects of finite precision arithmetic. Fortunately, it was found that the direct calculation of the eigenvectors from the EEG covariance matrix encountered no numerical conditioning problems.

As a final note, the KLE can be used to assist in the analysis of the EEG [3] [4]. The analysis of the EEG is equivalent to the analysis of a set of basis vectors and their associated weighting vectors. However, analyzing the KLE representation of the data introduces a major problem since the basis vectors are data dependent. Thus, the significance of the basis vectors and the associated weights must both be considered. Attempts to interpret the significance of the basis vectors is wrought with difficulty [5]. An additional problem with the KLE is that it not directly related to potential theory, thus further complicating the interpretation of the results.

7.1.3 Experimental Results

The experimental data consisted of the two 45 epoch data sets recorded for the eyes closed (EC) and eyes positioned (EP) conditions, as described in chapter 2. Recall that the data was magnitude and phase corrected and lowpass filtered. The sum of squares of each left-ear, average or perimeter referenced epoch was then normalized to a constant value.

Basis vectors were formed by sampling the spherical harmonic functions, as explained in chapter 3, using the electrodes' estimated spherical coordinates obtained in chapter 6. Optimal basis vectors for each data set were also determined by calculating the PCs from an averaged covariance matrix. The averaged covariance matrices were formed by adding together the covariance matrices obtained from each epoch in the data set. Plots of the sampled spherical harmonic functions and the PCs obtained from the left-ear referenced data sets are shown in Figures 7.1-7.3.

Reduced rank projections of the data sets based on the sampled spherical harmonic functions and the PCs were calculated. Since the results from the previous chapter indicated that a 4-th degree spherical harmonic expansion overfits the data, only the spherical harmonic functions associated with degrees of expansion ranging from $N = 0$ to 3 were considered. The corresponding rank ($P = (N + 1)^2$) projections obtained from the PCs were also determined.

The normalized residual variance E^2 , defined in (6.4.1), were calculated from the reduced rank projections, with the mean and standard deviation values shown in Tables 7.1 and 7.2 for both the left-ear and perimeter referenced data sets respectively. The normalized residual channel variances E_c^2 , defined by (6.4.2), for the 1st and 2nd degree spherical harmonic expansions of the perimeter referenced data sets are shown in Tables 7.3-7.6 and the associated Figures 7.4-7.7. Temporal PSDF estimates based on the residual signal obtained from the spherical harmonic expansion of the data sets are shown in Figure 7.8.

$N(P)$	EC Data Set		EP Data Set	
	SHE	KLE	SHE	KLE
0 (1)	51 $\pm$ 11	46.2 $\pm$ 8.4	39.0 $\pm$ 9.1	35.9 $\pm$ 8.2
1 (4)	12.0 $\pm$ 3.9	8.6 $\pm$ 1.8	9.5 $\pm$ 2.6	8.7 $\pm$ 2.2
2 (9)	4.3 $\pm$ 1.5	2.4 $\pm$ 0.6	3.5 $\pm$ 1.1	3.1 $\pm$ 1.0
3 (16)	1.8 $\pm$ 0.6	0.69 $\pm$ 0.22	1.5 $\pm$ 0.6	1.1 $\pm$ 0.5

Table 7.1 - Mean and standard deviation of $E^2 \times 100$ for a degree N spherical harmonic expansion (SHE) and a rank P KLE projection based on the left-ear referenced data sets.

$N(P)$	EC Data Set		EP Data Set	
	SHE	KLE	SHE	KLE
0 (1)	90.9 $\pm$ 2.6	50.5 $\pm$ 9.8	88.8 $\pm$ 1.7	60.5 $\pm$ 5.8
1 (4)	21.2 $\pm$ 4.0	12.5 $\pm$ 3.3	21.6 $\pm$ 2.7	16.0 $\pm$ 2.7
2 (9)	7.5 $\pm$ 1.7	3.70 $\pm$ 1.18	8.0 $\pm$ 1.6	5.9 $\pm$ 1.5
3 (16)	3.2 $\pm$ 0.7	1.14 $\pm$ 0.49	3.6 $\pm$ 1.0	2.21 $\pm$ 0.74

Table 7.2 - Mean and standard deviation of $E^2 \times 100$ for a degree N spherical harmonic expansion and a rank P KLE projection based on the perimeter referenced data sets.

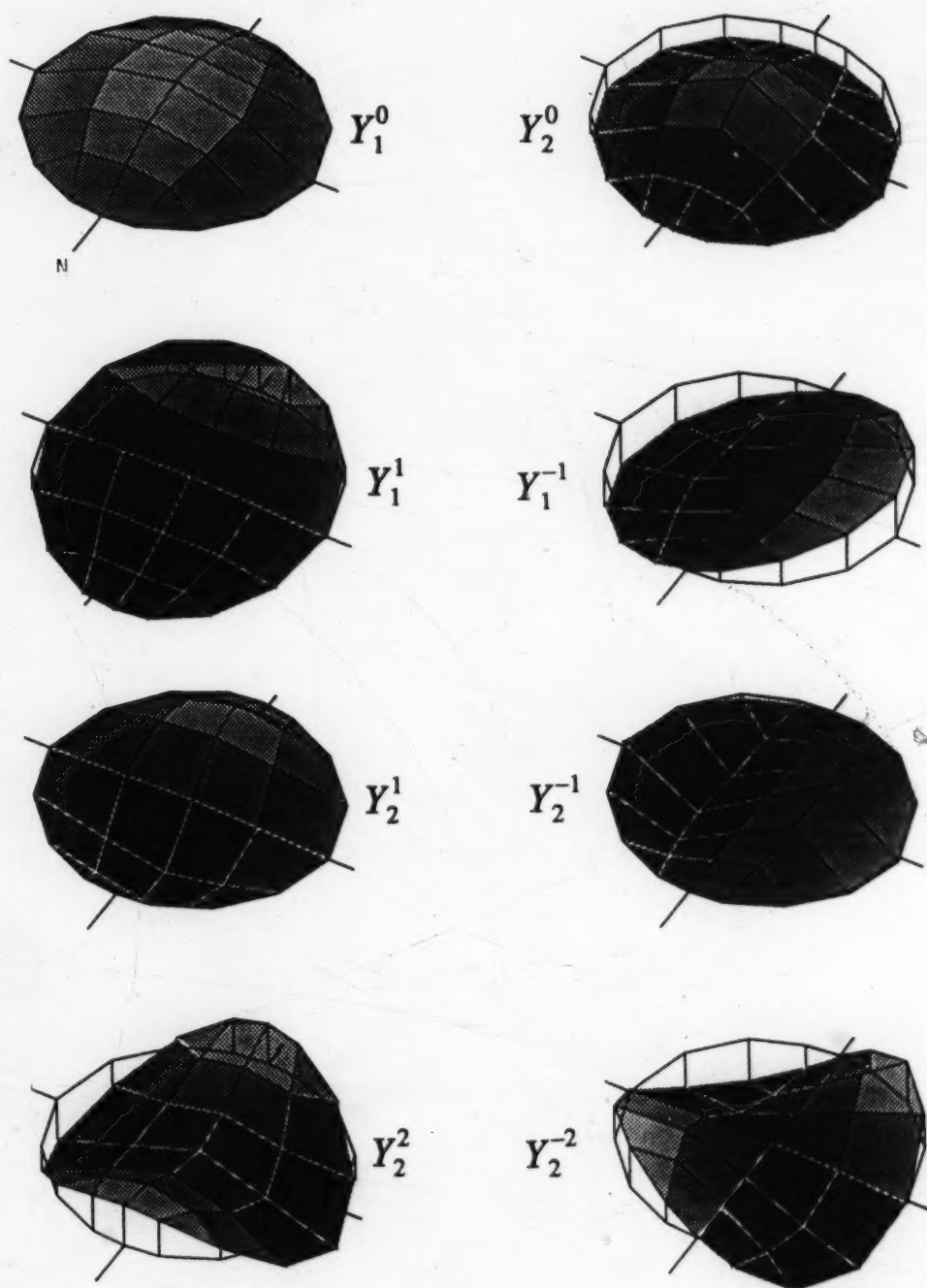


Figure 7.1 - Plot of the sampled spherical harmonic functions. The symbol 'N' represents the approximate position of the nasion.

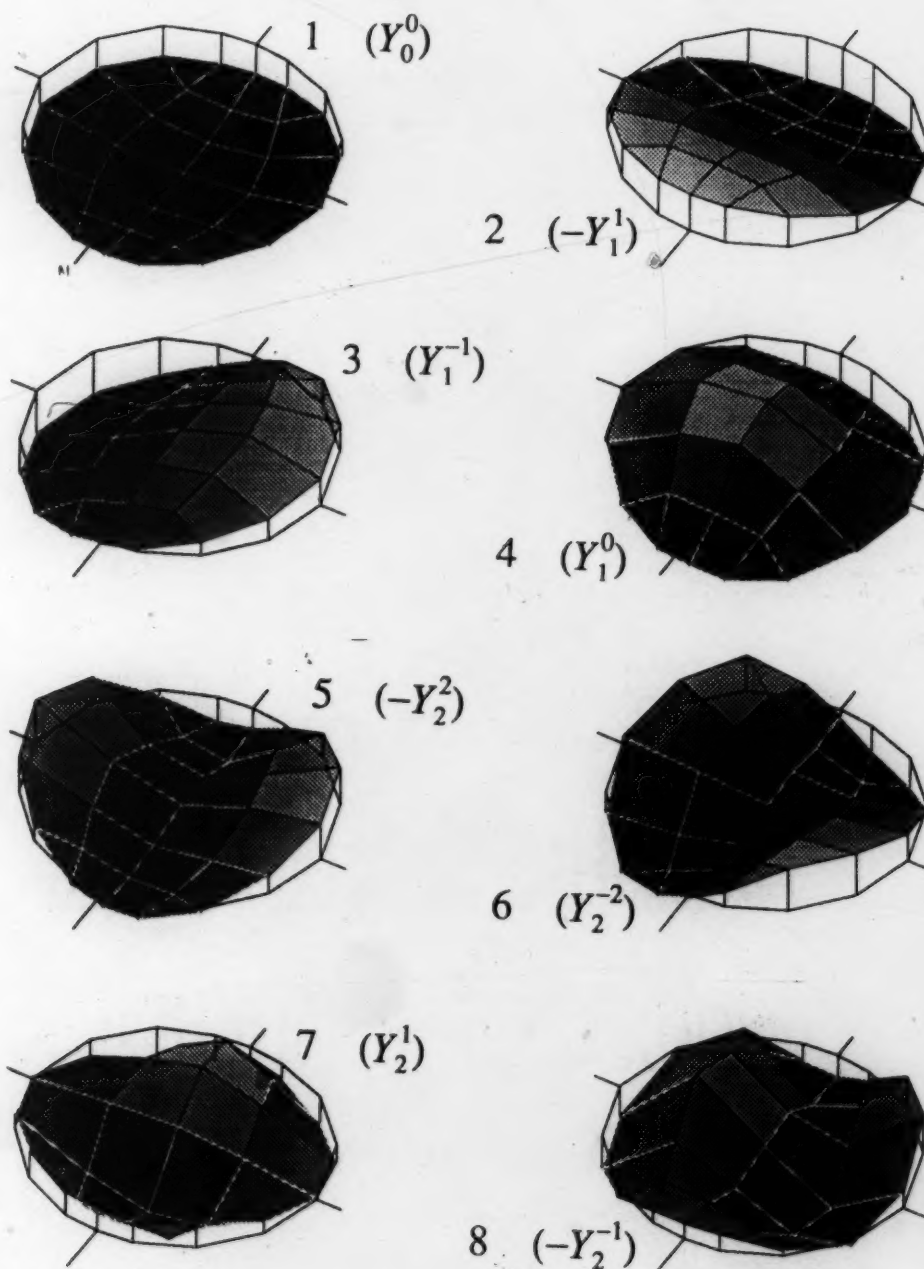


Figure 7.2 - The eight most significant PCs obtained from the left-ear referenced EP data set. The plots are numbered to indicate the significance of the PCs, with the first PC accounting for the largest percentage of the data set's variance. Shown in brackets is the sampled spherical harmonic function most similar to each of the PCs. Note that some of the PCs are inverted when compared to the sampled spherical harmonic functions.

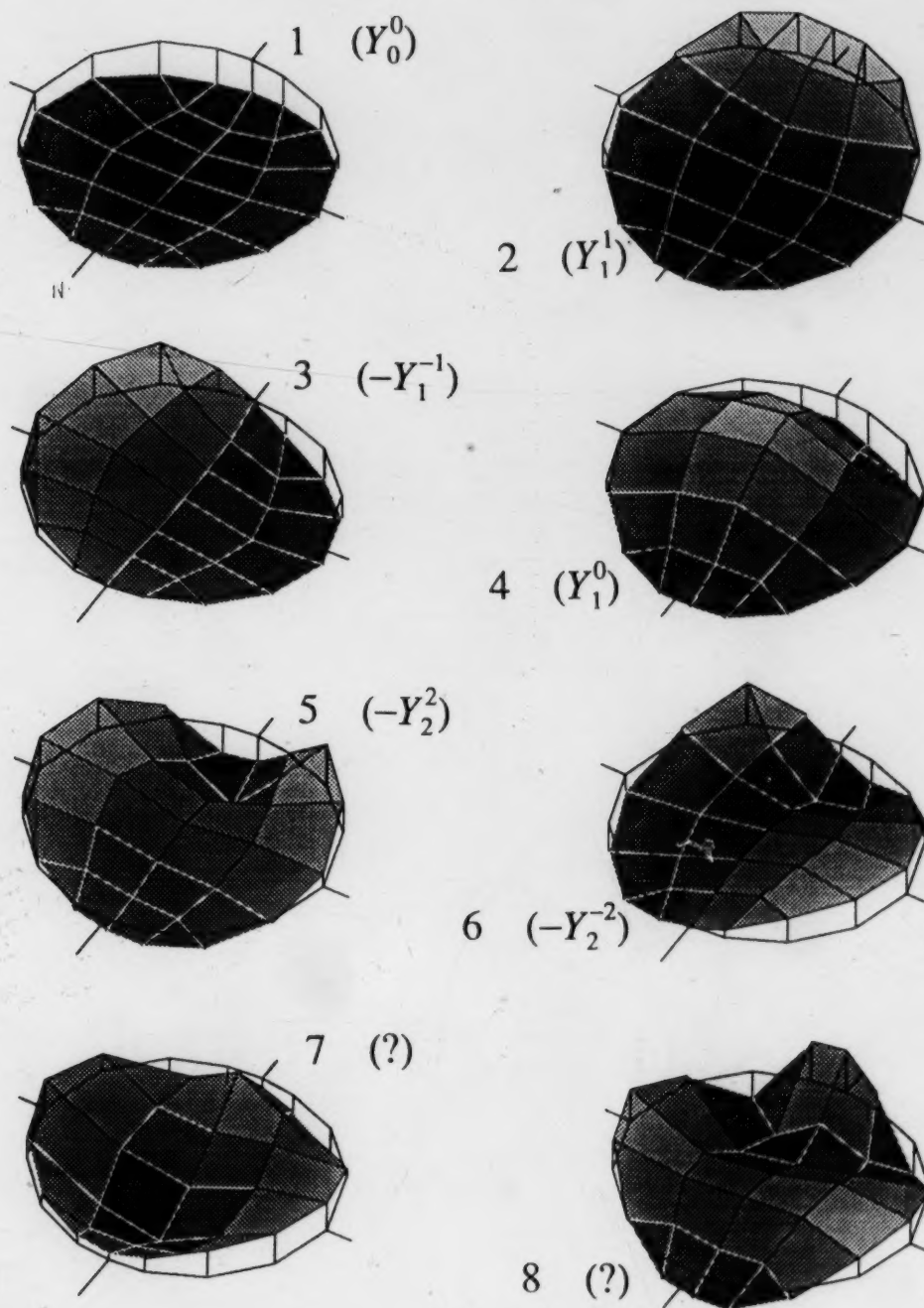


Figure 7.3 - The eight most significant principal components obtained from the left-ear referenced EC data set. The sampled spherical harmonic function most similar to each of the basis vectors is shown in brackets. Note that some of the principal components are inverted when compared to the sampled spherical harmonic functions.

Nasion

	0.41 ± 0.14	0.38 ± 0.11	0.41 ± 0.13	
0.51 ± 0.18	0.32 ± 0.10	0.34 ± 0.13	0.31 ± 0.11	0.54 ± 0.27
0.58 ± 0.15	0.39 ± 0.14	0.36 ± 0.19	0.37 ± 0.14	0.70 ± 0.20
0.80 ± 0.17	0.49 ± 0.14	0.29 ± 0.11	0.53 ± 0.18	0.92 ± 0.21
0.83 ± 0.33	0.37 ± 0.13	0.41 ± 0.15	0.40 ± 0.18	0.95 ± 0.38
1.47 ± 0.88	0.86 ± 0.47	1.49 ± 1.46	0.95 ± 0.57	1.80 ± 0.93
	0.98 ± 0.64	0.73 ± 0.26	1.31 ± 1.07	

Table 7.3 - Mean value and standard deviation of $E_i^2 \times 100$ for a 1st degree spherical harmonic expansion of the EC perimeter referenced data set.

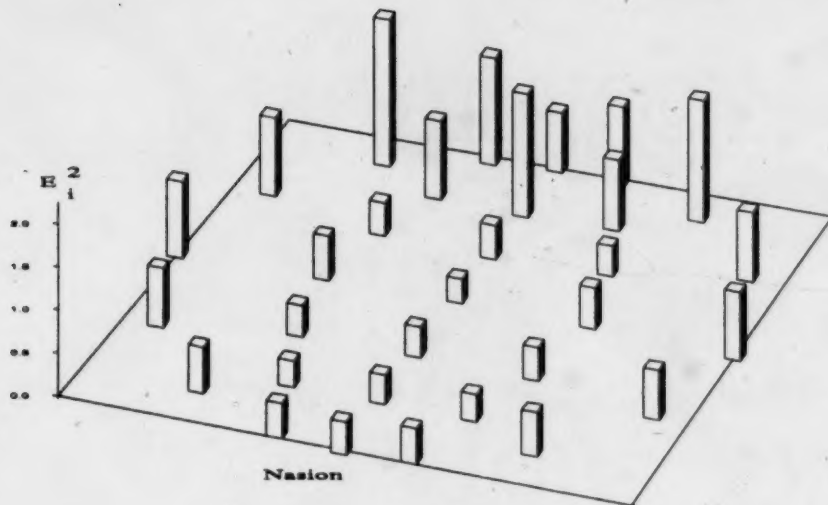


Figure 7.4 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 7.3.

	0.12 ± 0.06	0.11 ± 0.05	0.13 ± 0.07	
0.20 ± 0.07	0.10 ± 0.03	0.12 ± 0.08	0.11 ± 0.04	0.22 ± 0.13
0.17 ± 0.07	0.11 ± 0.06	0.13 ± 0.05	0.11 ± 0.05	0.19 ± 0.07
0.23 ± 0.08	0.17 ± 0.06	0.04 ± 0.01	0.18 ± 0.07	0.25 ± 0.10
0.27 ± 0.11	0.15 ± 0.06	0.14 ± 0.07	0.17 ± 0.12	0.35 ± 0.14
0.58 ± 0.41	0.24 ± 0.12	0.61 ± 0.70	0.27 ± 0.19	0.72 ± 0.39
	0.40 ± 0.24	0.40 ± 0.17	0.51 ± 0.44	

Table 7.4 - Mean value and standard deviation of $E_i^2 \times 100$ for a 2nd degree spherical harmonic expansion of the EC perimeter referenced data set.

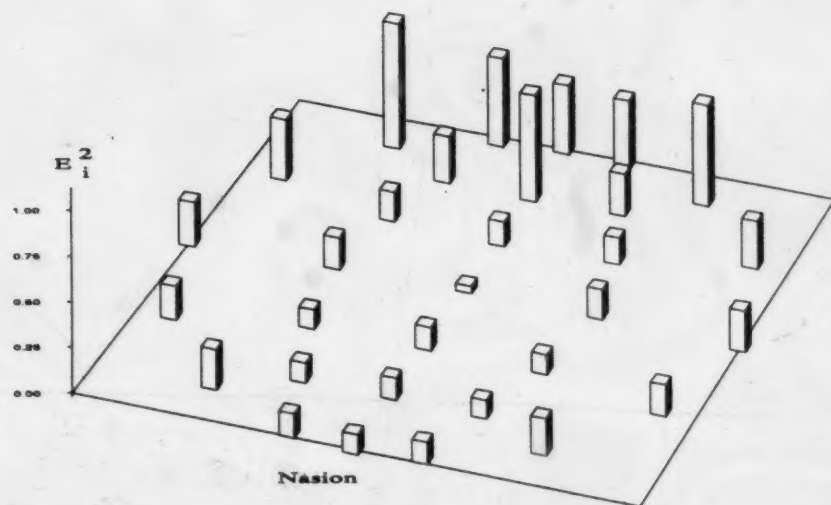


Figure 7.5 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 7.4.

	0.64 ± 0.18	0.55 ± 0.17	0.64 ± 0.27	
0.92 ± 0.40	0.49 ± 0.15	0.48 ± 0.17	0.54 ± 0.23	0.82 ± 0.32
0.87 ± 0.25	0.60 ± 0.25	0.63 ± 0.21	0.56 ± 0.21	0.84 ± 0.19
1.03 ± 0.37	0.70 ± 0.53	0.40 ± 0.18	0.70 ± 0.40	1.11 ± 0.94
0.84 ± 0.30	0.45 ± 0.10	0.41 ± 0.13	0.50 ± 0.23	0.79 ± 0.16
0.91 ± 0.31	0.59 ± 0.21	0.90 ± 0.58	0.66 ± 0.28	0.97 ± 0.33
	0.67 ± 0.20	0.62 ± 0.18	0.80 ± 0.44	

Table 7.5 - Mean value and standard deviation of $E_i^2 \times 100$ for a 1st degree spherical harmonic expansion of the EP perimeter referenced data set.

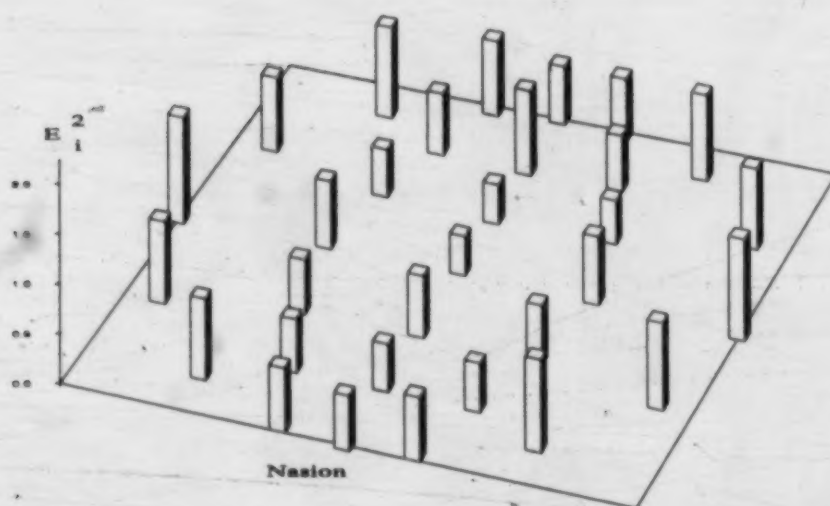


Figure 7.6 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 7.5.

	0.24 ± 0.09	0.22 ± 0.10	0.26 ± 0.17	
0.45 ± 0.28	0.16 ± 0.07	0.21 ± 0.08	0.20 ± 0.16	0.38 ± 0.20
0.34 ± 0.15	0.19 ± 0.08	0.22 ± 0.07	0.19 ± 0.10	0.32 ± 0.13
0.34 ± 0.22	0.26 ± 0.18	0.07 ± 0.02	0.25 ± 0.12	0.37 ± 0.51
0.31 ± 0.20	0.15 ± 0.04	0.15 ± 0.04	0.14 ± 0.05	0.31 ± 0.09
0.35 ± 0.11	0.19 ± 0.08	0.37 ± 0.26	0.23 ± 0.10	0.41 ± 0.16
	0.24 ± 0.08	0.24 ± 0.07	0.29 ± 0.19	

Table 7.6 - Mean value and standard deviation of $E_i^2 \times 100$ for a 2nd degree spherical harmonic expansion of the EP perimeter referenced data set.

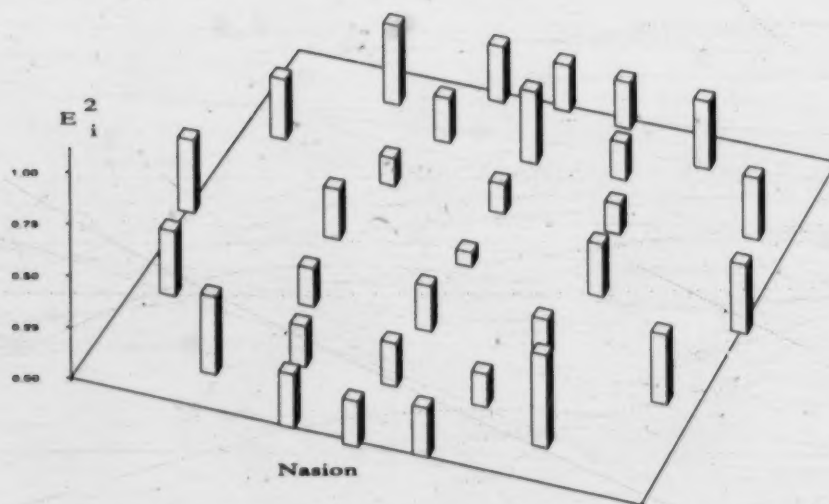
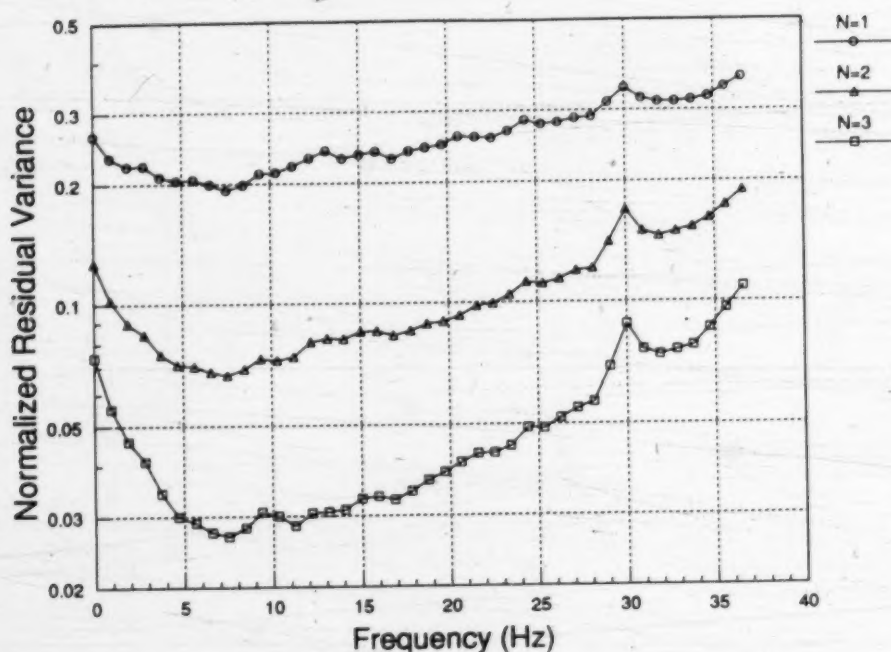
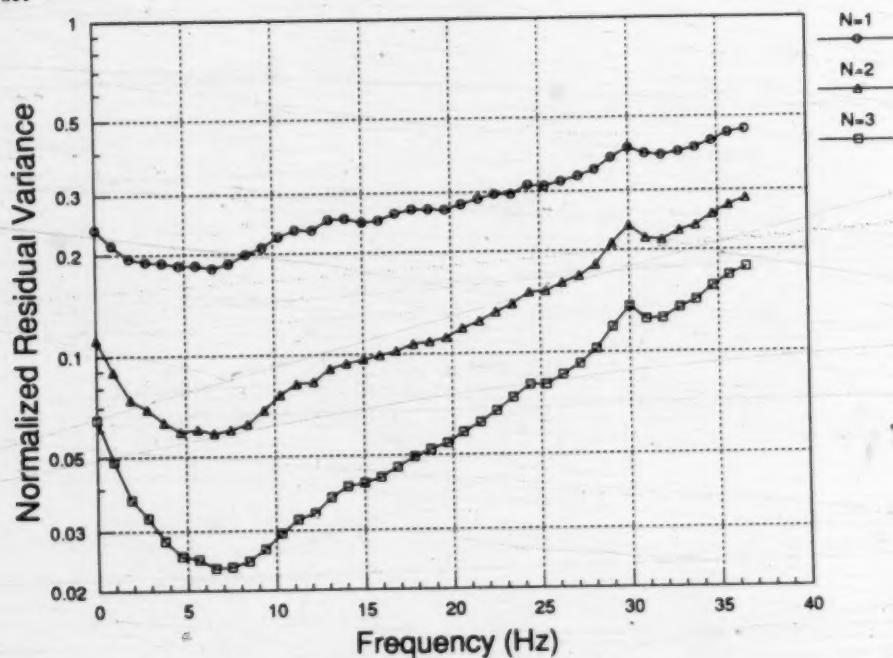


Figure 7.7 - Spatial plot of the mean values of $E_i^2 \times 100$ shown in Table 7.6.



(a) EC data set



(b) EP data set

Figure 7.8 - Normalized residual variance vs. frequency for the perimeter referenced data sets for various spherical harmonic expansion degrees. The main feature of the results is the global minimum in the variance at approximately 7 Hz which indicates the region of smallest spatial bandwidth. The peak in the residual spectrum at 30 Hz. is believed to be an equipment induced artifact.

7.1.4 Discussion of Results

The reduced rank KLE finds the optimal rank P projection of the data set, thus defining a lower bound for E^2 obtainable from a linear model with P degrees of freedom. The spherical harmonic expansion and the KLE of the left-ear referenced data sets produced similar results, especially for the EP data set, as shown in Table 7.1. Further similarity between the spherical harmonic expansion and the KLE of the EEG is illustrated by plots of the sampled spherical harmonic functions (Figure 7.1) and of the PCs obtained from the EP data set (Figure 7.2).

It is interesting to speculate as to why the sampled spherical harmonic functions are similar to the PCs. Since the skull tends to act as a spatial low-pass filter to signals originating from within the brain, one would expect the most significant PCs to be associated with low spatial frequencies. The specific spatial distribution of these low spatial frequencies depends upon the geometry of the problem. As mentioned before, one can approximate the exterior of the human head by the surface of a sphere, which represents a closed surface. Since the optimal basis vectors are orthogonal along with the fact that they must map onto the surface of the sphere, one would expect 'waves' (spherical harmonics) appear in the solution. The fact that the scalp's surface is only approximately spherical and that the data is observed on only half the sphere most likely accounts for the slight differences between the spherical harmonic expansion and the KLE.

There is less similarity between the spherical harmonic expansion and KLE results shown in Table 7.2 for the perimeter referenced EP data set. This result is due to the removal of the spatially constant values from the EEG by the perimeter reference. Since the degree N spherical harmonic expansion model includes a spatially constant basis vector Y_0^0 , the effective rank of the model is reduced by one to $(N + 1)^2 - 1$ for the perimeter referenced data. A similar problem does not exist for the KLE since the PCs are calculated directly from the data. Since the effective rank of the spherical harmonic expansion is less than the reduced rank KLE, the former's E^2 will increase, as the results indicate.

There is also less similarity between the spherical harmonic expansion and KLE results for the EC data set shown in Tables 7.1 and 7.2 and Figure 7.3. Again, although the specific reason for this difference is unknown, the following possible explanation can be presented. Comparing the KLE results for the perimeter referenced EC and EP data sets (Table 7.2) shows that E^2 is smaller for the EC data set, indicating more similarity between the epochs in the EC data set than in the EP data set. Recall that the EC data set has significant alpha activity, which is associated with synchronized widespread activity centered in the occipital region of the scalp. Due to the similarity of the alpha activity between individuals and due to the fact that the PCs are data dependent, the KLE can represent this activity by a reduced set of common basis vectors. In contrast, the EP data set had suppressed alpha activity and does not contain any known dominate spatial patterns. Since the sampled spherical harmonic basis vectors are independent of the data, only the PC method can take advantage of the similarity between the epochs in the EC data set due to alpha activity.

The E_i^2 results for the spherical harmonic expansion of the EC data set, shown in Tables 7.3-7.4 and the associated Figures 7.4-7.5, indicates that the largest residual variances occur in the occipital region, with a slight emphasis on the right side. Since the largest channel variances occur in this region, as illustrated in Figure 4.6 in chapter 4, one would also expect that E_i^2 should be related to the channel's variance. In addition, as previously noted, it is well-known that alpha activity is strongest in the occipital region. It would also be expected that the largest errors will occur for those channels nearest the source of the activity. As a first approximation, the spatial distribution of the alpha activity can be

represented by a radially orientated dipole located inside a spherical model of the head, directed towards the occipital region. The curvature of the potential function is greatest at approximately the point on the surface of the sphere pointed to by the dipole. It is reasonable to expect that E_i^2 will be greatest where the curvature is also the greatest.

For the E_i^2 results based on the perimeter referenced EP data set shown in Tables 7.5-7.6 and Figures 7.6-7.7, one can note a more even distribution of the variances as compared to the EC data set. This result is probably due to the more even spatial distribution of the EEG activity for the EP data set as discussed in chapter 4.

In both data sets one can note an increase in E_i^2 for the perimeter channels located near the ear for both data sets while the smallest values occurs for the centrally located channels. For any spatially sampled data set, one can more accurately fit a function through centrally located samples than perimeter located samples since perimeter samples do not have adjacent samples on one side. In addition, the perimeter electrodes are the most widely spaced along with being most prone to having localized muscle artifacts.

Spectral analysis of the residual signal obtained from the spherical harmonic expansion of the EEG (Figure 7.8) produced results similar to those obtained for the residual signal investigated in chapters 5 and 6. Again, the main feature in the spectrums is a global minimum in the residual variance around 7 Hz which indicates that the spatial bandwidth of the signal is smallest in that region. In addition, the residual variance increases at a faster rate for the EP data set than the EC data set as the frequency is increased beyond 7 Hz, indicating that the spatial bandwidth of the EP data set is higher than the spatial bandwidth of the EC data set.

It is possible to improve the spherical harmonic model by optimizing the positions of the electrodes with respect to E^2 . That is, an optimal set of electrode positions could be determined that minimizes E^2 with respect to the electrode positions. Another improvement could be the use of an ellipsoidal model for the head along with the use of ellipsoidal harmonic functions. However, based on the similarity between the spherical harmonic expansion and KLE results, especially for the EP data set, it was felt that the spherical harmonic model as used was adequate.

7.2 Analysis of the Spherical Harmonic Coefficient Sequences

The results from the previous section indicated that a low-degree spherical harmonic expansion extracts most of the variance in the EEG. This section further tests the validity of the spherical harmonic expansion by comparing the variances of the spherical harmonic coefficient sequences to the power of the original data. Ideally, the variance of the spherical harmonic coefficients is proportional to the variance of the original data. However, the spherical approximation of the head along with the hemisphere sampling could result in poor estimates for the spherical harmonic coefficients. In addition, temporal and spatial PSDF estimates based on the spherical harmonic coefficient sequences are obtained.

7.2.1 Experimental Results

The variance of the coefficient sequences for spherical harmonic expansion degrees of $N = 1$ to 3 were determined for each perimeter referenced epoch in the EC and EP data sets. The mean value and standard deviation of the variances of the zero and first degree coefficient sequences for each data set are shown in Table 7.7.

Spatial and temporal PSDF estimates were obtained from the coefficient sequences. Spatial PSDF estimates, as described in chapter 3, of each perimeter and average referenced epoch was obtained by

$$S_n = \sum_{i=1}^{N_i} \sum_{m=-n}^n [a_n^m(i)]^2 \quad 0 \leq n \leq N \quad (7.2.1)$$

where S_n is the power of the degree n terms and $a_n^m(i)$ is the spherical harmonic coefficient for temporal sample i . The mean value and standard deviation of S_n for the perimeter and average referenced data sets are shown in Tables 7.8 and 7.9 respectively. Temporal PSDF estimates based on the coefficient sequences obtained from a first degree expansion are shown in Figure 7.9. The spectral estimates based on each coefficient sequence was estimated using Welch's method as described in chapter 6.

7.2.2 Discussion of Results

Ideally, the estimates of the spherical harmonic coefficients and hence the variances of the sequences should be independent of the degree of expansion. However, this independence is violated due to the fact that the sampled spherical harmonic function are not orthogonal. The theoretical results contained in chapter 3 showed that when the value of a function is known only over the surface of a hemisphere, the solution for the non-zero order coefficients is independent of the degree of expansion. In contrast, the solution for the zero order spherical harmonic coefficients for a degree N expansion involves N coupled equations. Thus, the estimates for the zero order spherical harmonic coefficients will be affected by the value of N .

Low degrees of expansion should produce the best estimates of the spherical harmonic coefficients. As N increases, the effect of errors in the spherical approximation of the head and in the estimates of the spherical coordinates of the electrodes becomes more

significant. In other words, as more degrees of freedom are incorporated into the model, the probability of overfitting increases which would result in unreliable estimates for the spherical harmonic coefficients.

The most significant feature of the results shown in Table 7.7 is the large variance of the zero-order terms a_0^0 and a_1^0 for a 3rd degree expansion, as compared to the results obtained from lower degrees of expansion. This feature indicates that the third degree expansion is overfitting the data and should not be used. The results also show that the variances of a_0^0 and a_1^0 differ slightly for the 1st and 2nd degree expansions, while the variances of the non-zero order coefficients a_1^1 and a_1^{-1} are more similar. Again the non-orthogonality of the sampled zero-order spherical harmonic functions are the cause of this result, as explained previously.

Another observation can be made for the variances of a_0^0 , which relates to the spatially constant values in the data. The coefficient a_0^0 is directly related to reference level in the data since the activity of the reference represents spatially constant shifts in the data. Theoretically, neutral referenced data should result in a_0^0 equalling zero. Recall that the EEG data sets were perimeter referenced which eliminated the effects of the left-ear reference. The variances of a_0^0 are close to zero for $N = 1$ while the results for $N = 2$ are significantly larger. One possible explanation for this increased variance is that the 2nd degree expansion is slightly overfitting the data. However, this explanation is not supported by the cross-validated model results from the previous chapter. A more likely explanation lies in the assumption that the perimeter reference eliminates the spatially constant term from the data. Actually, the perimeter reference method only eliminates the effect of the left-ear reference and not necessarily the spatially constant term. One could equally argue that a 2nd degree expansion of the data could be used to estimate and hence eliminate the spatially constant values from the data.

As shown in Table 7.8, slightly different spatial spectral estimates are obtained for the perimeter referenced data sets, depending upon the degree of expansion. The most useful information is obtained from the 2nd degree expansion due the inclusion of the extra term S_2 . An indication of the rolloff rate of the spatial spectrum is given by the ratio S_2/S_1 , with smaller values indicating a faster rolloff and thus a smaller spatial bandwidth. For the EC and EP data sets, this ratio is 0.252 and 0.284 respectively, indicating that the EC data set has a smaller spatial bandwidth. This result is consistent with the results contained in chapter 5 obtained from the circular array of electrodes.

It was shown in (3.2.12) that there is a direct relationship between the integrated squared spherical function and the sum of the squares of the spherical harmonic coefficients of the function. For spherically sampled functions, this relationship can be approximated by (refer to appendix C for details)

$$\frac{4\pi}{N_s} \sum_{n=1}^{N_s} v_n^2(i) \geq \sum_{n=0}^N \sum_{m=-n}^n [a_n^m(i)]^2 \quad (7.2.2)$$

Thus, the variance of the data should be proportional to the sum of the variances of the spherical harmonic coefficient sequences. Since the variance of each epoch was normalized to unity and $N_s = 31$, the variance of the coefficient sequences should be less than $4\pi/N_s \approx 0.6$.

The results contained in the last column in Table 7.8 show that the sum of the variances of the coefficient sequences for $N=1$ approximately equals 0.6, while the variances for $N=2$ are significantly larger. This observation indicates a slight overfitting of the data by a 2nd degree expansion.

Previous results have indicated that the perimeter reference provides a better estimate of the reference than the average reference. The superiority of the perimeter reference can be further illustrated by comparing the S_0 values obtained from the perimeter and average referenced data sets shown respectively in Tables 7.8 and 7.9. Note that S_0 is equivalent to the power of the a_0^0 coefficient, which in turn is related to the reference level in the data. It can be observed that the estimates for S_0 are smaller for the perimeter vs. average referenced data sets, again indicating the superiority of the perimeter reference.

Temporal PSDF estimates based on the coefficient sequences obtained from first degree spherical harmonic expansions of the perimeter referenced data sets are shown in Figure 7.9. The spectral plots obtained from the EC data set show a peak at approximately 10 Hz. while the spectral plots obtained from both data sets generally showed a rolloff with increasing frequency. These characteristics are similar to the composite PSDF estimates of the EEG data sets shown in Figure 6.13. It can also be noted that the general shape of the first degree coefficients' spectrums are similar. What these results do not show are the relationships between the coefficient sequences, which is the subject of the next section.

	EC Data Set			EP Data Set		
	N=1	N=2	N=3	N=1	N=2	N=3
a_0^0	0.005 ± 0.001	0.167 ± 0.065	3.55 ± 1.24	0.004 ± 0.001	0.208 ± 0.063	4.00 ± 0.99
a_1^0	0.306 ± 0.094	0.539 ± 0.172	5.69 ± 2.05	0.396 ± 0.063	0.602 ± 0.129	6.64 ± 1.71
a_1^1	0.173 ± 0.031	0.175 ± 0.032	0.130 ± 0.032	0.142 ± 0.020	0.140 ± 0.025	0.136 ± 0.033
a_1^{-1}	0.063 ± 0.018	0.063 ± 0.019	0.163 ± 0.068	0.076 ± 0.013	0.077 ± 0.015	0.231 ± 0.077

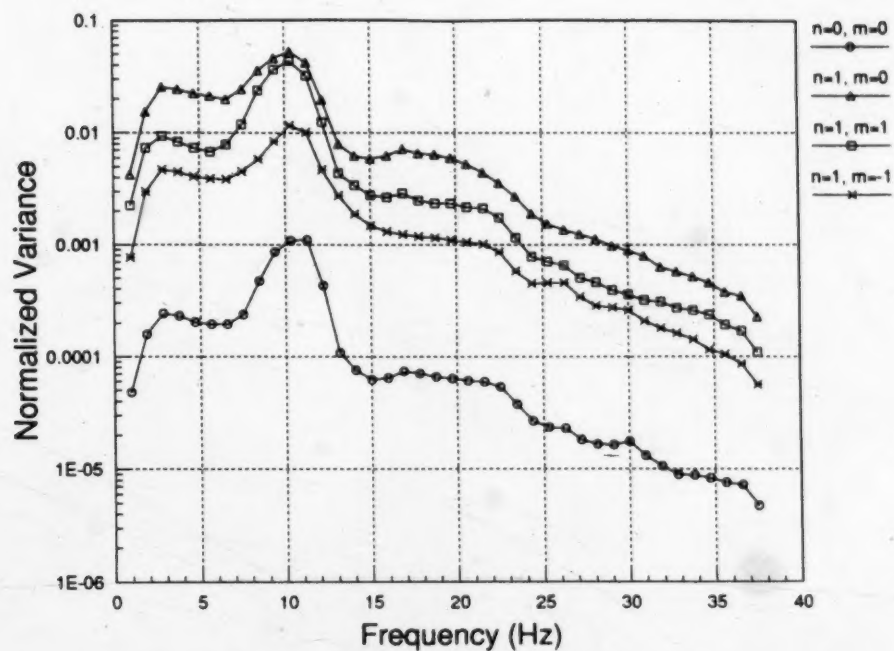
Table 7.7 - Mean value and standard deviation of the variances of the spherical harmonic coefficient sequences obtained from the perimeter referenced data sets.

	n=0	n=1	n=2	Sum
EC N=1	0.004 ± 0.001	0.543 ± 0.079	-----	0.547
N=2	0.167 ± 0.065	0.778 ± 0.158	0.196 ± 0.054	1.141
EP N=1	0.004 ± 0.001	0.615 ± 0.051	-----	0.619
N=2	0.208 ± 0.063	0.819 ± 0.115	0.233 ± 0.057	1.260

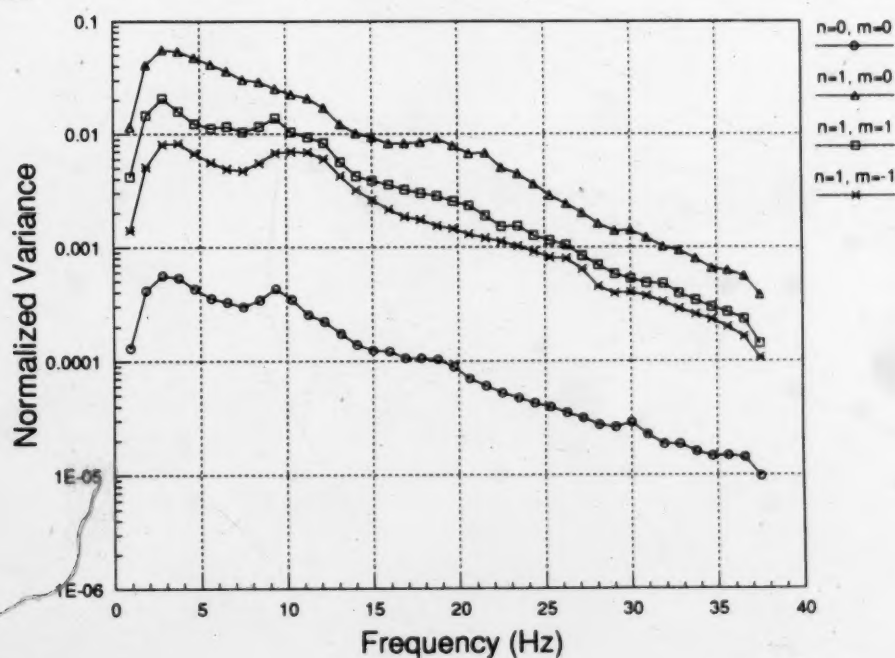
Table 7.8 - Spatial spectral (S_n) estimates based on spherical harmonic expansions (N=1 and 2) of the perimeter referenced data sets. The results shown in the last column are the sums of the enters in each row.

	n=0	n=1	n=2	Sum
EC N=1	0.107 ± 0.035	0.600 ± 0.103	-----	0.707
N=2	0.306 ± 0.104	0.860 ± 0.194	0.217 ± 0.064	1.383
EP N=1	0.139 ± 0.025	0.694 ± 0.070	-----	0.833
N=2	0.339 ± 0.078	0.924 ± 0.137	0.263 ± 0.066	1.526

Table 7.9 - Spatial spectral (S_n) estimates based on 1st and 2nd degree spherical harmonic expansions of the average referenced data sets. Note the increase in the power of S_0 in relationship to the results for the perimeter referenced data sets which indicates that the average reference provides a poorer estimate of the reference level than the perimeter reference.



(a) EC data set



(b) EP data set

Figure 7.9 - Temporal PSDF estimates based on the spherical harmonic coefficient sequences (a_n^m) obtained from a first degree expansion of the perimeter referenced data sets. The relatively small spectral values for the a_0^0 sequences are due to the use of the perimeter reference.

7.3 Lissajous Trajectories of the Spherical Harmonic Coefficient Sequences

The previous section investigated the variances of the individual spherical harmonic coefficient sequences. This approach was not particularly insightful since no particular significance can be attached to any single spherical harmonic coefficient, with the possible exception of the a_0^0 term which includes the effects of the active reference. A better approach is to perform a multivariate analysis of the spherical harmonic coefficient sequences.

One multivariate analysis method that has been used on the EEG is the plotting and segmentation of Lissajous trajectories [6] of the 3-channel auditory brain-stem response (ABR). The 3-channel ABR was differentially recorded using 3 electrode pairs placed on opposite sides of the head, namely front-back, left-right and top-lower back pairs. A 3-D coordinate system was formed by passing each axis through a different electrode pair. The coordinate system along with measured data was transformed into an orthogonal coordinate system, and the transformed data plotted as a Lissajous diagram. The evolution of the trajectories over time was investigated and the results showed the trajectories formed consistent and often planar paths.

The validity of this approach is based on the small size and central location of the brain-stem which allows its electrical activity to be represented by a centrally located dipole. It was shown that the coordinate axis in the 3-CLT are equivalent to the coordinate strengths of a centrally located dipole. Thus, the 3-CLT is a convenient method of visualizing the ABR since the 3-CLT provides sufficient information to uniquely define the dipole's parameters and since the dipole model is a reasonable representation of the ABR.

However, it is inappropriate to model the background EEG by a dipole placed in the central region of the brain, for higher brain functioning is not located in that region. This however does not rule out the use of a dipole model to describe the surface potential, for a infinite number of internal generator configurations can result in the same surface potential. An alternate interpretation of the dipole parameters is that they describe strength and orientation of a spherical wave of wavenumber index one (one cycle of the wave around the head). As explained in chapter three, the three first-degree spherical harmonic coefficients describe the coordinate strengths of a centrally located dipole. Thus, the coordinate axis of the 3-CLT can also be considered as crude estimates of the first degree spherical harmonic coefficients. Due to the usefulness of the 3-CLT in investigating the behavior of the brain, the three first degree coefficients were analyzed by plotting their Lissajous trajectories.

The purpose of this section is to perform a preliminary analysis of the EEG data sets via the Lissajous trajectories of the first degree spherical harmonic coefficient sequences, which will be referred to as the CSLT (Coefficient Sequence Lissajous Trajectory). Both the strength and the weakness of plotting the trajectories is that no statistical assumptions are made about the nature of the data. Most statistical analysis techniques, such as spectral analysis, assume that the signal is stationary and linear. It is well-known that the EEG is at best stationary for only short time intervals while the linearity of the EEG is largely unknown. The disadvantage of investigating the trajectories is that they require visual inspection, with the amount of information extracted dependent upon the expertise of the investigator.

7.3.1 Experimental Results

As explained previously, the first degree spherical harmonic coefficients can be considered as the coordinate strengths (x, y, z) of a spherical wave of wavenumber index one, where

$$x = a_1^{-1} \quad y = -a_1^1 \quad z = a_1^0 \quad (7.3.1)$$

while the magnitude R of the wave is given by

$$R = \sqrt{(a_1^0)^2 + (a_1^1)^2 + (a_1^{-1})^2} \quad (7.3.2)$$

Each set of spatial samples in the EEG produces one set of first-degree spherical harmonic coefficients which, in turn, define the xyz-coordinates of one point in the Lissajous diagram. Lissajous trajectories are formed by connecting successive points in time.

Results presented in the previous section indicated that the most reliable estimates of the first degree spherical harmonic coefficients were obtained by a first degree expansion. Thus, the coefficient sequences were determined from the 1st degree spherical harmonic expansion of the EEG data. Analysis of each EEG epoch produced 960 point coefficient sequences, resulting in one 960 point CSLT.

The CSLTs were analyzed with the aid of a computer graphics program that allowed the evolution of the trajectories to be viewed in an interactive environment and included the ability to rotate the coordinate axis. Special interest was paid towards EEG epochs with significant alpha activity. Visual inspection of the trajectories showed that planar elliptical orbits about the origin often formed during periods of strong alpha activity. These elliptical trajectories would last for several cycles of the alpha activity, with the entering and exiting phases of the trajectories having sharp transitions. When the alpha rhythm was suppressed, no visible structure in the trajectories was evident.

In order to illustrate the above observations, one EC epoch, plotted in Figure 7.10a, was selected for further analysis. It can be noted that the epoch has strong alpha activity located in the occipital region of the head with several short sessions of frontal alpha activity. Alpha activity was emphasized by digitally bandpass filtering each channel from 8 to 13 Hz., with the results shown in Figure 7.10b. The first degree spherical harmonic expansion of the filtered EEG was determined, with the resulting fit shown in Figure 7.11b. Plotted in Figure 7.12 are the three first-degree spherical harmonic coefficient sequences along with the magnitude R of the spherical wave. The CSLT during a period of strong alpha activity is shown in Figure 7.13. By rotating the coordinate system of Figure 7.13, Figure 7.14 illustrates that the trajectory formed a planar elliptical segment. The sharp transitions of the entering and exiting phases of the elliptical trajectory are shown in Figure 7.15. A sequence of spatial plots, for the elliptical trajectory time period, of the original and filtered EEG along with the first degree spherical harmonic expansion fit are shown in Figures 7.16-18 respectively. As a final example, a linear Lissajous trajectory is shown in Figure 7.19.

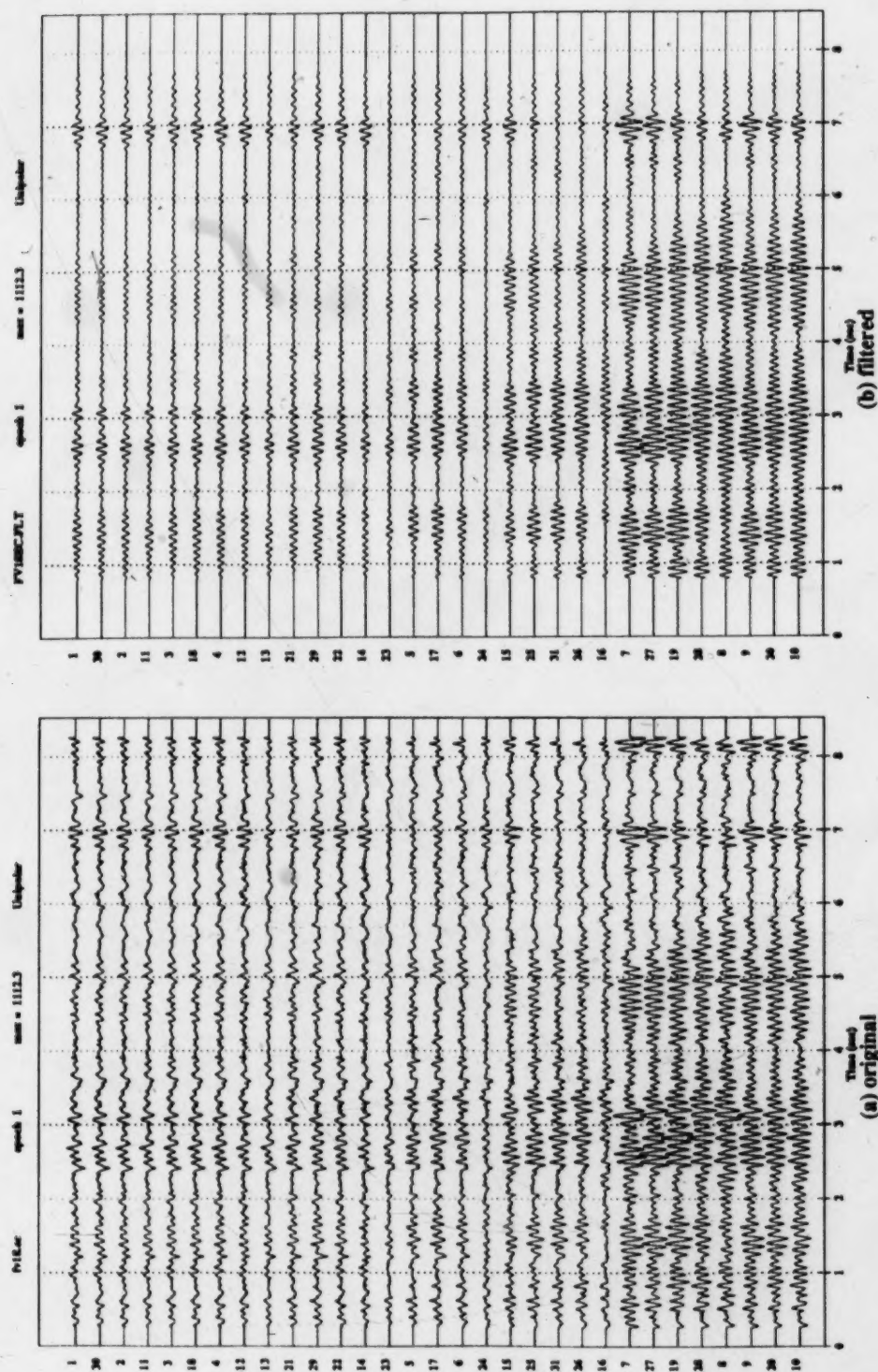
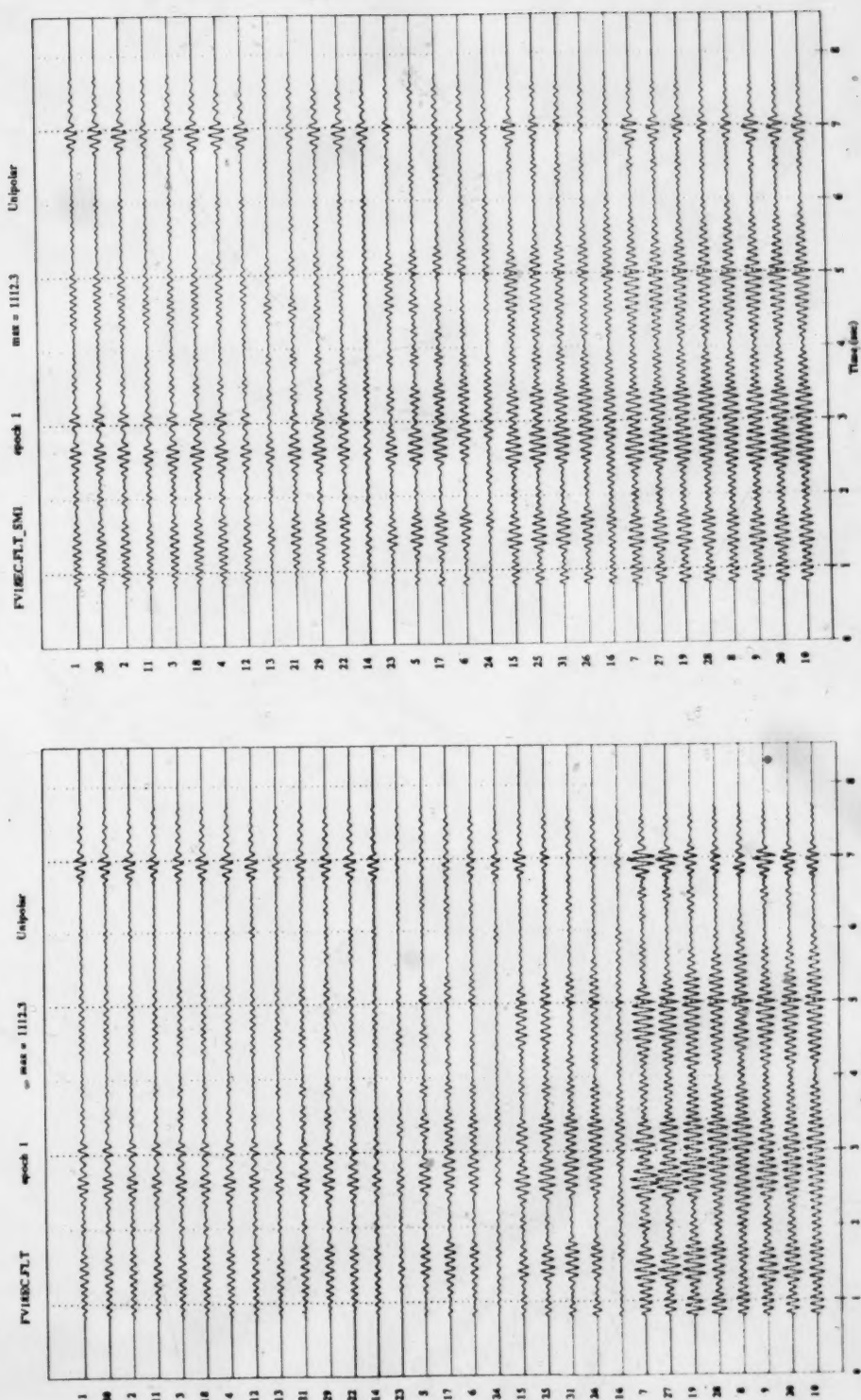


Figure 7.10 - Temporal plots of the original and bandpass filtered (8-13 Hz) EEG epoch. Channel numbers are indicated on the left-hand side of the plot, with the locations of the channels indicated in Figure 2.4. Note the strong alpha activity in the occipital region of the scalp.



(b) spherical harmonic expansion fit

(a) filtered

Figure 7.11 - Temporal plots of the filtered and first degree spherical harmonic fit of the filtered epoch.

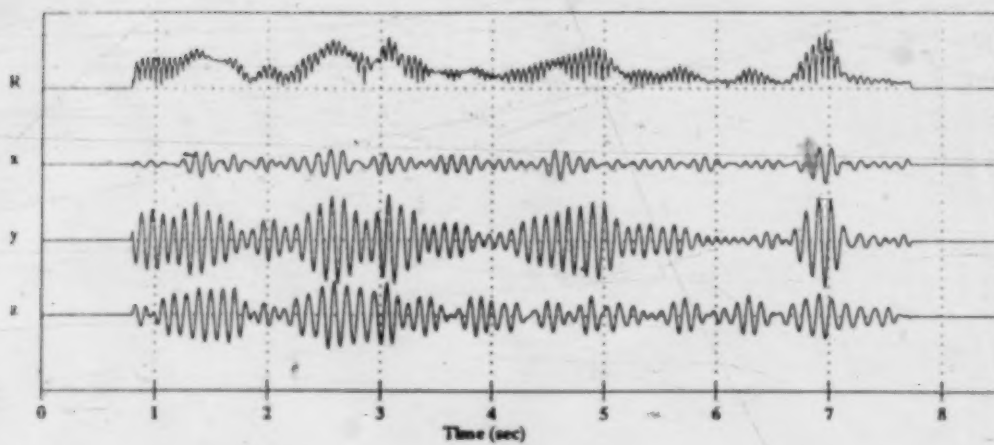


Figure 7.12 - Plot of the three first degree spherical harmonic coefficient sequences along with the associated strength R .

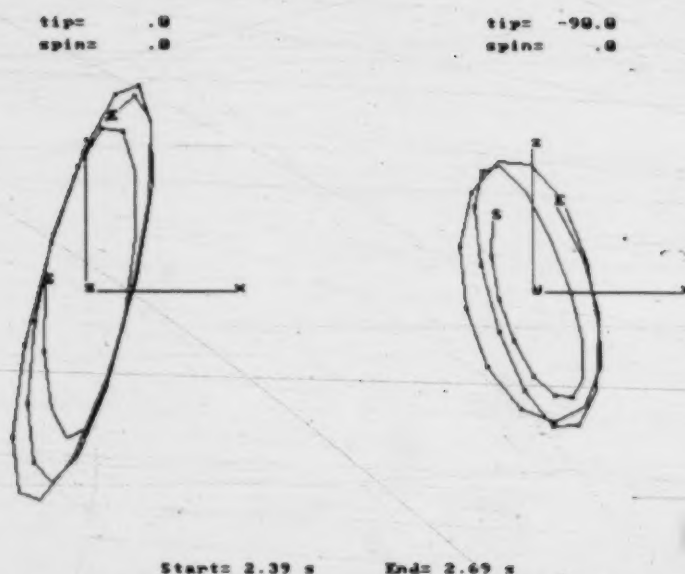


Figure 7.13- Example of an elliptical CSLT. The left plot is a top view of the trajectories, with the +z axis pointing towards the viewer. The right plot is obtained by rotating the left plot -90 degrees about the x-axis, resulting in the y-axis pointing into the page. The symbols 'S' and 'E' represent the starting and ending points respectively of the trajectory.

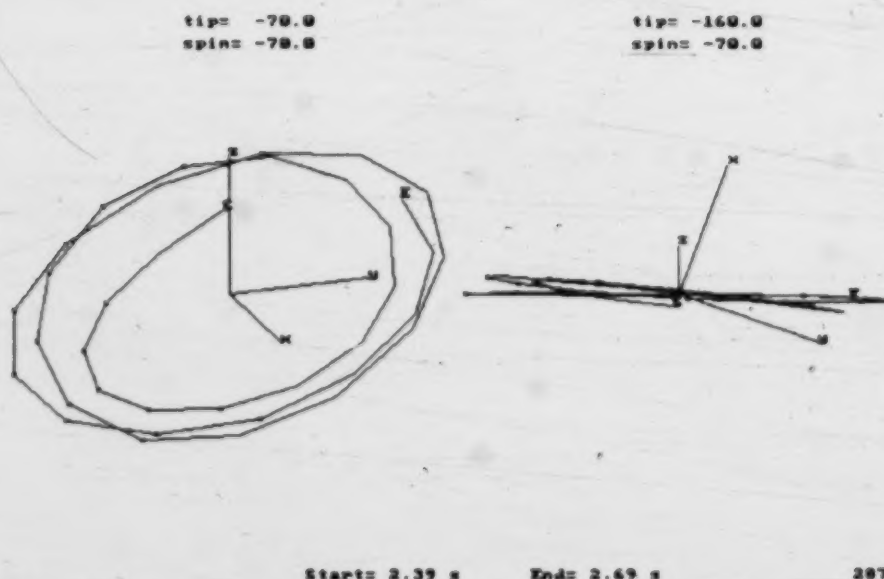
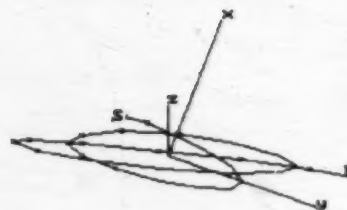
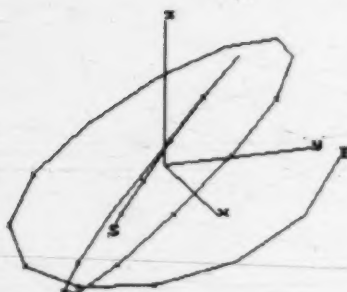


Figure 7.14 - Rotated version of the CSLT shown in Figure 7.13. The coordinate system of the left plot was rotated until the trajectory plane was perpendicular to the viewer. The right plot then views the trajectory plane on edge.

tip= -70.0
spin= -70.0

tip= -160.0
spin= -70.0



Start= 2.22 s

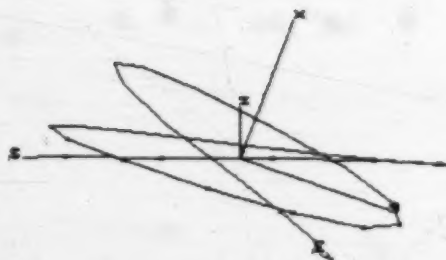
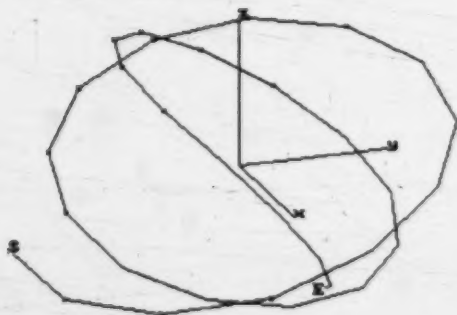
End= 2.47 s

266 296

(a) entering phase

tip= -70.0
spin= -70.0

tip= -160.0
spin= -70.0



Start= 2.63 s

End= 2.89 s

316 347

(b) exiting phase

Figure 7.15 - Lissajous trajectory of the (a) entering and (b) exiting phases of the elliptical section shown in Figure 7.14. Note that the elliptical section forms and dissipates in less than one cycle of the alpha rhythm (-12 points).

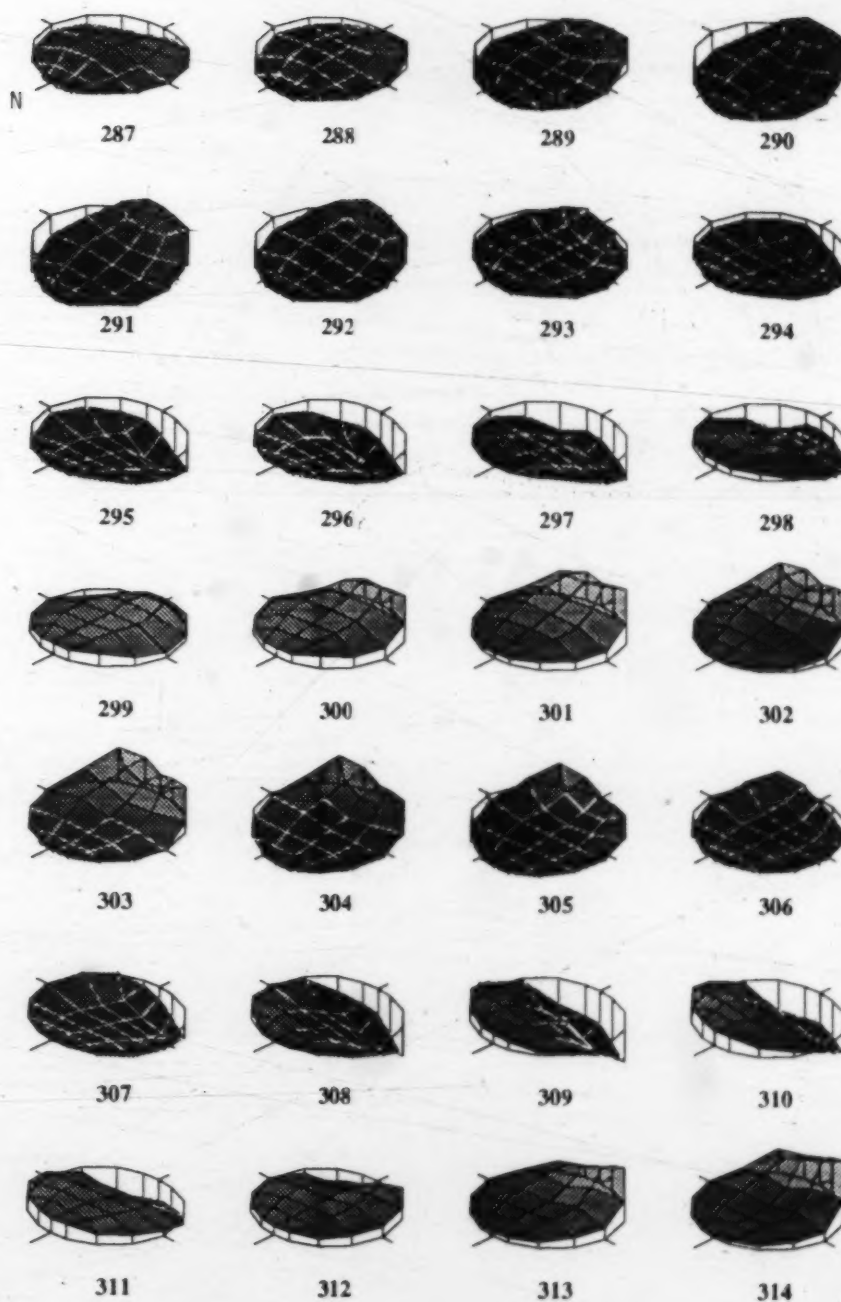


Figure 7.16 - Spatial plot of the left-ear referenced EEG epoch. Shown beside each plot is the sample number (120 samples/sec). Larger positive values are represented by lighter regions while the surrounding ring represents zero potential.

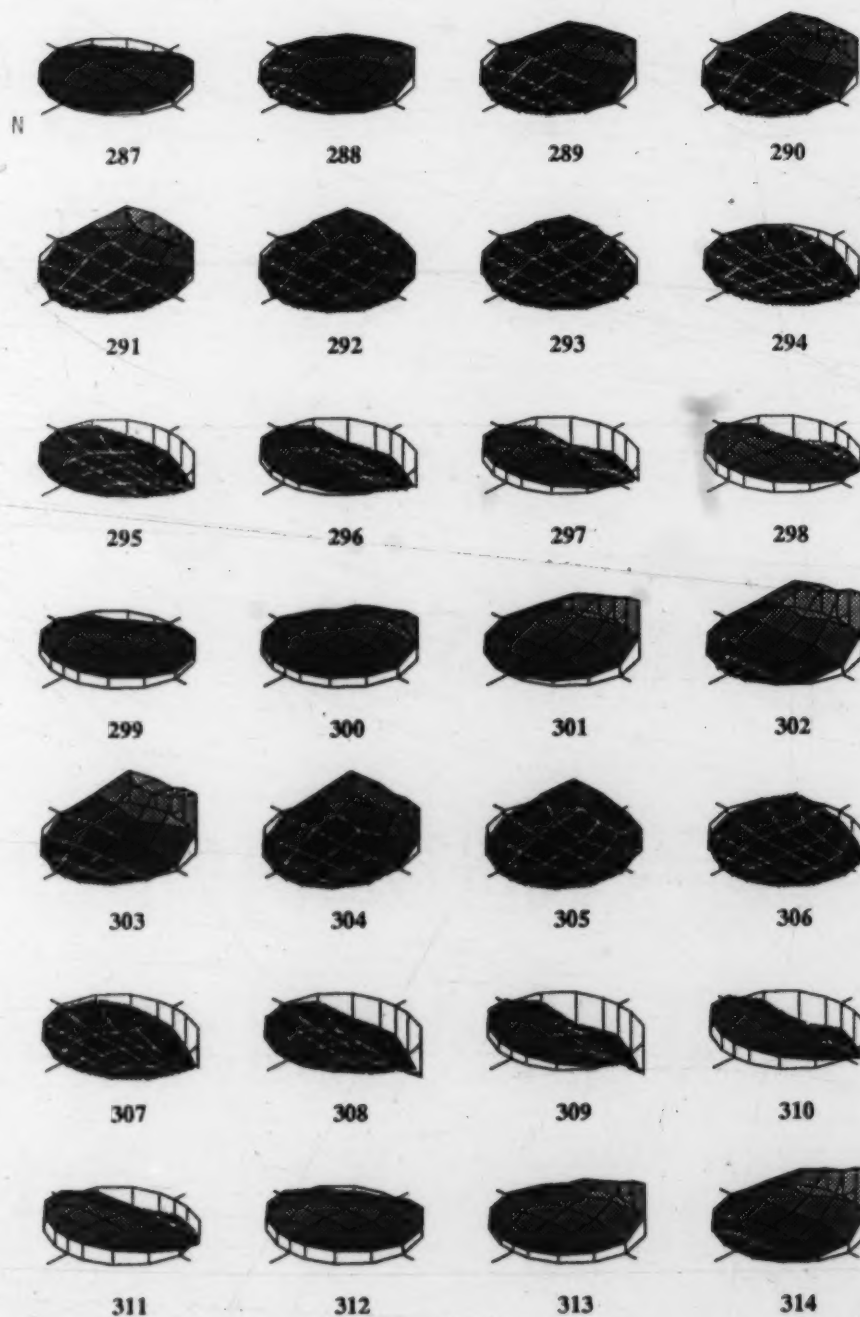


Figure 7.17 - Spatial plot of filtered EEG data. Each channel in the EEG shown in Figure 7.13a was digitally bandpass filtered from 8-13 Hz.

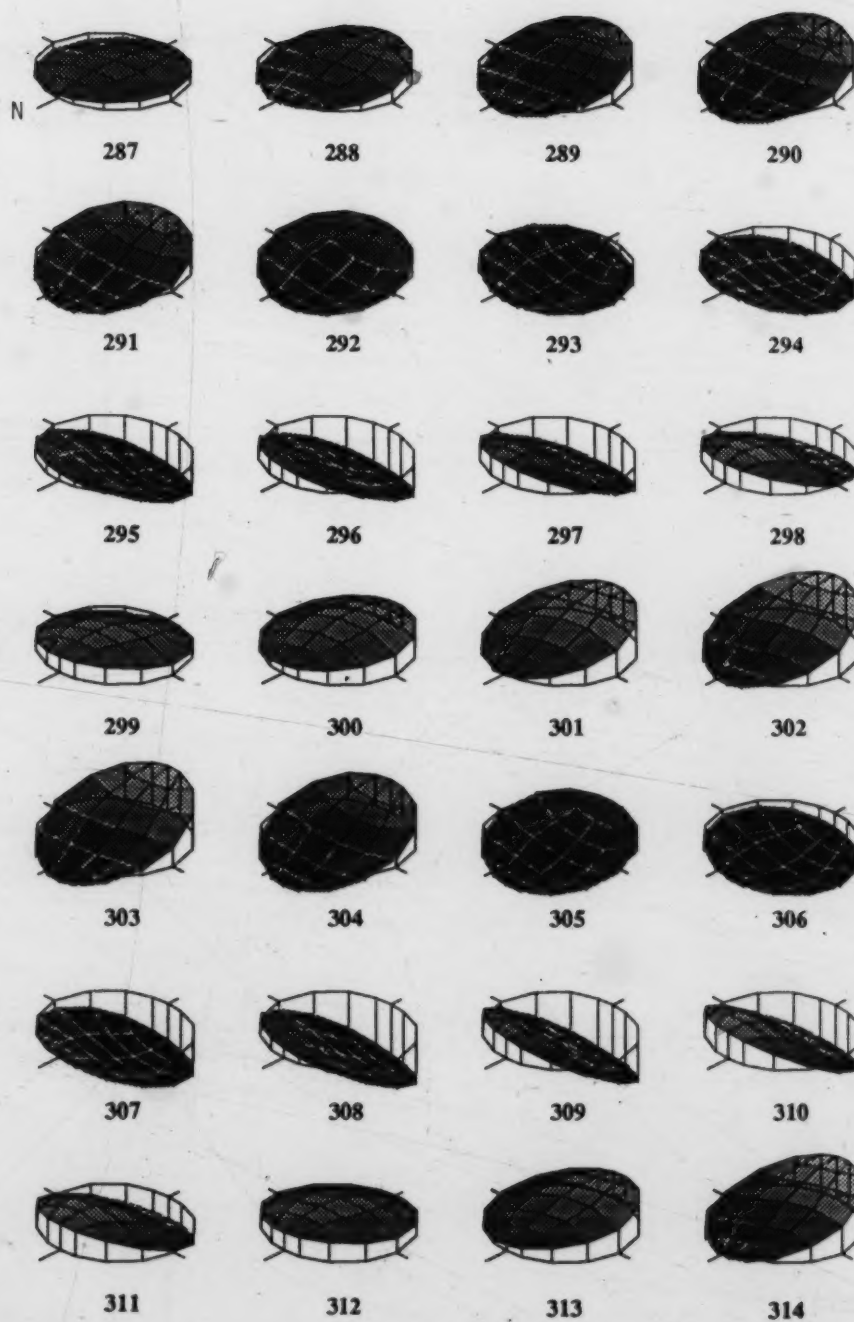


Figure 7.18 - Spatial plot of the 1st degree spherical harmonic expansion fit of the filtered EEG epoch.

tip= .0
spin= .0

tip= -90.0
spin= .0



Start= 6.78 s

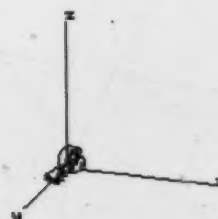
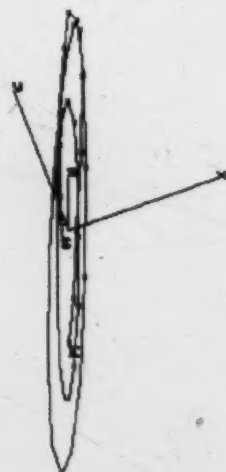
End= 7.00 s

813 850

(a) unrotated

tip= -20.0
spin= 20.0

tip= -110.0
spin= 20.0



Start= 6.78 s

End= 7.00 s

813 850

(b) rotated

Figure 7.19 - Example of a linear CSLT for (a) an unrotated and (b) a rotated coordinate system. The right coordinate system in (b) was rotated until the linear segment was pointing towards the viewer. The left plot thus shows a side view of the linear segment.

7.3.2 Discussion of Results

The experimental results indicated that the CSLTs sometimes formed either linear or elliptical planar paths during periods of strong alpha activity. One interpretation of this result is that either a standing or a travelling spherical potential wave sometimes forms on the exterior of the cerebral cortex during periods of strong alpha activity. It can be easily shown that the theoretical CSLT obtained for a standing spherical wave will form a linear trajectory, with the trajectories distance from origin indicating the strength of the wave. Due to the varying thickness of the skull, the amplitude of a spherical wave travelling around the outer surface of the cerebral cortex will vary when the potential is measured on the scalp and result in an approximately circular CSLT. When the plane of the circular CSLT encompasses the origin, then the wave travels along the equivalent of a circumferential path on the surface of the sphere.

Global theories relating to the generation of the EEG, as discussed by Nunez [1] [2] [3], predict the existence of standing and traveling spatial waves in the background EEG, although quantitative verification has been limited. Special interest has been applied to the alpha rhythm due to the relatively widespread spatial extent of the activity. The temporal period of the alpha activity would then be related to the propagation velocity of the wave along with the circumference of the outer surface of the brain. The experimental results contained in this section are consistent with the existence of spatial waves in a portion of the alpha rhythm of the EEG.

It should be noted that although the CSLT involves only the first degree spherical harmonic coefficients, the trajectories can still be used to track the movement of a spherical wave that is composed of many wavenumbers. Since the orientation of the spherical harmonic components in a spherical wave should rotate as a group, the first degree coefficients provide sufficient information to track the movement of the wave. However, when multiple spherical waves travel along different paths, the CSLT will track only the average position of the waves.

Although the results presented in this section are preliminary, it appears that part of the alpha rhythm in the EEG is composed of either standing or travelling spherical waves in the EEG. More importantly, the CSLT provides a simple method of detecting the presence of travelling or standing waves in the EEG.

7.4 Conclusions

It was shown that the sampled spherical harmonic functions were similar to the optimal basis vectors obtained via the KLE of the EEG. This suggests that the spherical harmonic expansion of the EEG is indeed an appropriate representation of the sampled signal. It was further shown that the most reliable estimates of the spherical harmonic coefficients were obtained for a first degree spherical harmonic expansion of the EEG.

Lissajous trajectories of the first degree spherical harmonic coefficients indicated that elliptical planar segments sometimes formed during periods of strong alpha activity. One explanation of this result is that part of the alpha rhythm consists of a spherical wave propagating around the outer surface of the cerebral cortex. However, due to the preliminary nature of the results, further research is need to verify this interpretation of the results.

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8 Summary and Conclusions

8.1 Summary of Results

The purpose of this research was to perform a spherical harmonic analysis of the EEG. The experimental data consisted of 15 subjects' EEGs recorded for the eyes closed (EC) and the eyes positioned (EP) conditions. The EEGs were sampled with a 31-channel data acquisition unit and later compensated for the magnitude and phase differences introduced by the unit between the channels.

Results have been obtained which showed that the sampled spherical harmonic functions were similar to the optimal basis vectors obtained via the Karhunen-Loève expansion (KLE). In addition, the spherical harmonic expansion of the EEG provided a good dimensionality reduction, with approximately 92% of the signal's variance accounted for by a 2nd degree expansion involving nine coefficients. These results encouraged further analysis of the spherical harmonic coefficient sequences obtained from the EEG. The validity of the spherical harmonic expansion was tested and it was shown that the first degree expansion provided the most reliable estimates of the spherical harmonic coefficients. However, if the purpose is only to achieve a dimensionality reduction with the spherical harmonic expansion, it was found that the optimal degree of expansion was two. The reason for these two different expansion degrees relates to the difficulty in estimating the zero order coefficients. Theoretical analysis of the spherical harmonic expansion of a function known only on the surface of a spherical cap showed that the solution for the zero order coefficients were coupled while the solution for the non-zero order coefficients remained uncoupled. This result is due to the fact that the spherical harmonic functions of zero order evaluated over the surface of a spherical cap are non-orthogonal while the non-zero order functions remain orthogonal. Sampling the spherical harmonic functions only increases this problem. Increasing the expansion degree involves additional zero order coefficients, resulting in an increased interaction between the solution for the zero order coefficients.

Preliminary analysis of the spherical harmonic coefficient sequences considered two approaches. For each data set, spectral analysis of the coefficient sequences obtained from a first degree expansion produced similar results and were also similar to the averaged temporal spectrums of the EEG. These results suggest that the coefficient sequences are not independent and should instead be investigated as a group. The relationships between the first degree spherical harmonic coefficient sequences were investigated by plotting their Lissajous trajectories. The three coefficients of first degree can be considered as the coordinate strengths of the spherical wave that has one cycle around the sphere (wavenumber index=1). The Lissajous trajectories illustrated that planar elliptical and planar linear trajectories often formed during periods of strong alpha activity. One possible interpretation of these planar trajectories is that a portion of the alpha activity consists of either traveling or standing waves.

One of the problems encountered with the analysis of the EEG is the activity of the common reference electrode. It was first assumed that the use of the average reference would eliminate this problem. Theoretical analysis and experimental results clearly illustrated a previously unknown problem with the average reference. It was shown that the average reference causes an unequal decrease in the channels' variance when compared to the channels' variance of the common reference data, with the largest decrease occurring for centrally located channels. The undesirable effect of using the average reference is that the variance of the perimeter channels appears to increase when compared to the variance of the central channels. In order to reduce this effect, a new reference, referred to as the perimeter reference, was presented that is based on the averaged activity of the perimeter electrodes. However, the reference problem is still not completely solved, for the theoretical

analysis suggested that all reference estimates based on linear transformations of the EEG will encounter problems. The experimental results simply suggest that the perimeter reference is a better estimate of the neutral reference than the average reference.

Another problem with performing a spatial analysis of the EEG is that the signal is not adequately spatially bandlimited by the recording hardware, inferring that some of the signal's variance will be spatially aliased. The adequacy of the spatial sampling by the 31-electrode montage was tested by estimating the spatially aliased proportion of the EEG's variance based on data recorded by a circular array of electrodes. The spatially aliased portion of the variance for the 31-channel EEG was estimated to be approximately 1% to 5%. It was shown that the spatially aliased variance was related to the temporal frequency, with minimal aliasing occurring from 5 to 12 Hz and with significantly increased spatial aliasing beyond 12 Hz. This later result also indicated a relationship between the spatial wavenumbers and temporal frequencies in the EEG, which in turn, is consistent with the existence of wave activity in the EEG. The experimental results indicated that the spatial aliasing of the EEG was limited to an acceptable level.

One application of a mathematical representation of the spatial EEG, such as a spherical harmonic expansion, is to produce topographic maps of the electrical activity. In order to determine the suitability using the spherical harmonic expansion model as a mapping method, the spherical spline and the bilinear model were also considered. By cross-validating the results, the adequacy of the models were tested without the need for extra recording channels. Results showed that the spherical spline provided a near optimal interpolating linear model for the spatial EEG, with the largest errors occurred for those channels located on the perimeter of the electrode montage. The smoothing spherical harmonic expansion model showed a significant increase in the error, as compared to the results obtained for the spherical spline. As expected, the bilinear model produced the largest errors of the three models. These results suggest that the spherical spline should be used for mapping noise free EEG data while the spherical harmonic expansion should be used to produce smoothed maps of noisy data. It was also shown that the optimal parameters for the spherical spline model were $\kappa = 2$ and $\lambda = 0$ while the optimal degree of the spherical harmonic expansion was $N = 2$.

In order to determine the optimality of the models in describing the spatial EEG, a multiple linear regression (MLR) model was also developed. The MLR model attempts to predict the activity of each channel based on a linear combination of the spatial activity of the remaining channels and provides the optimal cross-validated linear model. Since the MLR model is data dependent, estimates of the model's coefficients are affected by the activity of the reference electrode. It was shown that by imposing one linear constraint on the solution to the MLR model, the dependence of the model's coefficients on the reference activity is eliminated. The constrained MLR model also has the advantage that the residual signal is unaffected by the activity of the reference. Experimental results obtained from the constrained MLR model indicated that the spatially unpredictable portion of the signal's variance was approximately 5% and 9% respectively for the perimeter referenced EC and EP data sets. The constrained MLR model was modified to include reduced rank estimates which, in turn, allowed the spatial dimensionality of the EEG to be investigated. It was found that the vast majority of the predictable part of the spatial EEG could be accounted for by a model of rank 15-20.

8.2 Conclusions

The experimental results indicated that the spherical harmonic expansion of the EEG provided a good dimensionality reduction. It was found that the sampled spherical harmonic functions were similar to the optimal basis vectors obtained via the KLE. The validity of the spherical harmonic expansion was tested and it was shown that the first degree expansion provides the most reliable estimates of the spherical harmonic coefficients. Lissajous trajectories of the first degree spherical harmonic coefficient sequences illustrated that planar elliptical trajectories often formed during periods of strong alpha activity. One possible interpretation of these planar trajectories is that a portion of the alpha activity consists of either traveling or standing waves.

Analysis of the average reference showed that it produced a significantly larger decrease in the variance of central vs. perimeter channels for data that was recorded with respect to a common reference. The net effect of the average reference is that the variance of perimeter channels appears to increase relative to the variance of centrally located channels. It was shown that the perimeter reference introduced in this thesis reduced this effect, inferring that the perimeter reference provides a better estimate of the neutral reference than that provided by the average reference.

The spatially aliased portion of the EEG's variance for a 31-channel recording was estimated to be approximately 1% to 5%. It was shown that the spatially aliased variance was related to the temporal frequency, with minimal aliasing occurring from 5 to 12 Hz and with significantly increased spatial aliasing beyond 12 Hz. This later result also indicated a relationship between the spatial wavenumbers and temporal frequencies in the EEG which, in turn, is consistent with the existence of wave activity in the EEG. The experimental results indicated that the spatial aliasing of the EEG was limited to an acceptable level.

The quality of the spherical harmonic expansion maps were compared, using the method of cross-validation, to the maps produced by a bilinear and a spherical spline model. Optimality of the mapping methods was determined with the aid of the optimal reference independent linear model based on a constrained multiple linear regression approach. Results showed that the spherical spline model provided a near optimal linear model for the spatial EEG while the spherical harmonic expansion maps had a significantly increased error level. In addition, the optimal parameters for the spherical spline model were $\kappa = 2$ and $\lambda = 0$ while the optimal degree of truncation for the spherical harmonic expansion was $N = 2$.

In conclusion, the results presented in this thesis show that the spherical harmonic expansion of the EEG can be used to study the gross features of the signal. One particularly useful application of the expansion is to detect the presence of spatial wave activity in the EEG. Wave activity in the EEG has been proposed in the literature, but experimental verification has been limited. Several of the experimental results presented in this thesis are consistent with the existence of wave activity in a portion of the EEG.

8.3 Suggestions for Further Research

The results contained in this thesis suggest further research in the following areas.

1. An optimal set of spherical coordinates for the electrodes could be determined by minimizing the residual variance obtained from the spherical harmonic expansion of the EEG. A non-linear optimization algorithm would determine the electrodes' positions that produce the lowest residual variance for the EEG data sets. The spherical coordinates of the electrodes based on the measured positions could be used as a starting point. An optimal set of electrode coordinates might improve the coefficient estimates obtained from a second degree expansion.

2. Several methods of detecting spatial abnormalities in the EEG could be compared to determine their clinical suitability. The residual signal determined from the truncated spherical harmonic expansion and the reduced rank constrained MLR model emphasizes spatially localized activity. Both of these methods can be considered as spatial high-pass filters. The sharpness of the spatial filter obtained from the spherical harmonic expansion is controlled by the degree of expansion while the sharpness of the MLR model is controlled by the rank of the model. One additional method of detecting spatial abnormalities that could be considered is the spatial Laplacian of the EEG estimated from the spherical spline.

3. The detection of planar segments in the Lissajous trajectories of the first degree spherical harmonic coefficients could be automated. As discussed in the thesis, one interpretation for the existence of planar trajectories is that either travelling or standing waves are present in the EEG. The trajectories squared distance from the best fitting plane could be calculated for short segments of the EEG. Another approach that has been used in other work is to calculate the bending energy of the trajectories. The purpose of such an approach would be to determine a quantitative measure of the rate of occurrence of the planar trajectories.

4. It is possible to segment the EEG based on the bending energy of the Lissajous trajectories. Preliminary work (not shown in thesis) suggested that the trajectories formed periods of relatively constant curvature, separated by short transitions. Segmentation of the EEG by its spatial properties would allow detection of short period micro-states in the brain.

Appendix A - Legendre Polynomials and Associated Functions

Legendre polynomials form an orthogonal set of functions over the range $-1 \leq x \leq 1$ and can be generated via the following expression

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (A-1)$$

where n is the degree of the polynomial. It can be shown that $P_n(x)$ has n distinct zeros located in the range $-1 \leq x \leq 1$. Several of the lower degree polynomials are given by

$$P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{1}{2}(3x^2 - 1) \quad P_3(x) = \frac{1}{2}(5x^3 - 3x) \quad (A-2)$$

It is possible to recursively calculate the polynomials via

$$(n+1)P_{n+1}(x) - (2n+1)xP_n(x) + nP_{n-1}(x) = 0 \quad (A-3)$$

The associated Legendre functions of the first kind and of positive integral order can be calculated from the Legendre polynomials via

$$P_n^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m P_n(x)}{dx^m} \quad 0 \leq m \leq n \quad (A-4)$$

$$= 0 \quad m > n$$

where m is the order and n is degree of the function. With the aid of (A-1), the above expression can be written as

$$P_n^m = \frac{1}{2^n n!} (1-x^2)^{\frac{m}{2}} \frac{d^{n+m}}{dx^{n+m}} (x^2 - 1)^n \quad (A-5)$$

It should be noted that associated Legendre functions are often referred to as Ferrers associated Legendre functions $T_n^m(x)$. To confuse matters even worst, an alternate definition for $P_n^m(x)$ is sometimes used that adds a $(-1)^m$ factor onto (A-4), which then leads to equations with different signs. With the substitution $x = \cos \phi$, (A-4) can be written as

$$T_n^m(\cos \phi) = \sin^m \phi \left(\frac{d^m P_n(x)}{dx^m} \right)_{x=\cos \phi} \quad (A-6)$$

The associated functions can also be calculated recursively, for a varying order via

$$P_n^{m+1}(x) = \frac{2mx}{\sqrt{1-x^2}} P_n^m(x) - (n+m)(n-m+1)P_n^{m-1}(x) \quad (A-7)$$

and for a varying degree via

$$(n-m+1)P_{n+1}^m(x) = (2n+1)xP_n^m(x) - (n+m)P_{n-1}^m(x) \quad (A-8)$$

Appendix B - Spherical Harmonic Expansion of a Function Defined on the Surface of a Spherical Cap

This appendix determines the spherical harmonic expansion of a function defined over the surface of a spherical cap. Let E represent the surface integral of the squared difference between the function v and a spherical harmonic expansion.

$$E = \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} \left(v - \sum_{n=0}^{\infty} \sum_{m=-n}^n a_n^m Y_n^m \right)^2 \sin(\phi) d\phi d\theta \quad (B.1)$$

The partial derivative of E with respect to a_n^m is given by

$$\frac{\partial E}{\partial a_n^m} = \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} \left(-2v Y_n^m + 2Y_n^m \sum_{q=0}^{\infty} \sum_{p=-q}^q a_q^p Y_q^p \right) \sin(\phi) d\phi d\theta \quad (B.2)$$

Equating the above expression to zero, with some additional rearrangement, results in

$$\sum_{q=0}^{\infty} \sum_{p=-q}^q a_q^p \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} Y_q^p Y_n^m \sin(\phi) d\phi d\theta = \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} v Y_n^m \sin(\phi) d\phi d\theta \quad (B.3)$$

Using (3.3.1), the above equation can then be simplified to

$$\sum_{q=0}^{\infty} \sum_{p=-q}^q a_q^p R(m, n; p, q; \phi_0) = \int_{\phi=0}^{\phi_0} \int_{\theta=0}^{2\pi} v Y_n^m \sin(\phi) d\phi d\theta \quad (B.4)$$

Appendix C - Relationship Between the Sum of the Squares of the Spatial Samples and the Spherical Harmonic Coefficients

Assume that the value of a function is known on the surface of a sphere and that its spherical harmonic expansion is also known. As was shown in chapter 2, the relationship between the integrated squared function and the spherical harmonic coefficients is given by

$$\int_S v^2 ds = \sum_{n=0}^{\infty} \sum_{m=-n}^n (a_n^m)^2 \quad (C.1)$$

where the integral is evaluated over the surface of the sphere. If the function is approximated by a n -th degree spherical harmonic expansion, then

$$\int_S v^2 ds \geq \sum_{n=0}^N \sum_{m=-n}^n (a_n^m)^2 \quad (C.2)$$

Now assume that the function is sampled at N_s locations that are approximately uniformly distributed over the surface of the sphere. The integral in (C.2) can be approximated by dividing the sphere's surface into $4\pi/N_s$ regions of equal area as follows

$$\int_S v^2 ds = \frac{4\pi}{N_s} \sum_{i=1}^{N_s} v_i^2 \quad (C.3)$$

where v_i is the value of the function in the middle of the i -th region. Combining the two previous equations results in

$$\frac{4\pi}{N_s} \sum_{i=1}^{N_s} v_i^2 \geq \sum_{n=0}^N \sum_{m=-n}^n (a_n^m)^2 \quad (C.4)$$

Sampling the function over the surface of a hemisphere presents a problem, for the integration (or summation) should be performed over the whole surface of the sphere. The value of a squared function integrated over the surface of a hemisphere is approximately one-half the value obtained over the full surface of the sphere. However, each sample now represents one-half the area ($\frac{4\pi}{2N_s}$) as that represented by samples for full sphere sampling.

Thus, both of these factors cancel each other and (C.4) is still approximately correct for hemispherical sampling.